MATH 2113/CSCI 2113, Discrete Structures II

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Toby Kenney

Mock Midterm Examination

Calculators not permitted. Answers may be left in reasonably simplified forms – e.g. binomial coefficients, factorials, etc.

These questions should probably take more than an hour and a half. This is so that I can include a larger variety of questions, to give a better idea of the sorts of questions that might appear on the actual midterm.

Compulsory questions

1 Each course is graded by assigning one of the 11 grades A+, A, A-, B+, B, B-, C+, C, C-, D, or F to each student in the course. Suppose a given course has 15 students.

(a) In how many ways can grades be assigned to students in the course.

(b) Suppose the instructor also has to produce a summary stating how many of each grade have been given out, but not to which students. How many different summaries are possible?

2 Recall that a derangement of n elements is a permutation with no fixed points – i.e. $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that there are no values of m for which $\sigma(m) = m$. Use the inclusion-exclusion principle to find the number of derangements of an n-element set. [You can leave your answer expressed as a sum. Hint: use inclusion-exclusion to find the number of permutations with at least one fixed point.]

3 Define a recurrence by $a_n = 3a_{n-1} - 2a_{n-2} - 13n, a_0 = 0, a_1 = 4$.

(a) Find the generating function for the sequence a_n.

(b) Find a general formula for a_n.

4 (a) How many sets of 5 distinct cards from a standard deck contain an even number of kings?

(b) What is the probability that a 5 card poker hand is a full house given that it contains an even number of kings? [Recall that a full house is a hand with 3 cards of one rank and two cards of another.]

(c) What is the probability that a 5-card poker hand contains an even number of kings given that it is a full house?

5 (a) A five-card hand is dealt at random from a standard 52 card deck. What is the probability that it contains the king of hearts.

(b) A five-card hand is dealt at random from each of 5 standard decks (so 5 hands are dealt in total). What is the probability that one of the hands contains the king of hearts?

6 Show that $\sum_{k=0}^{m} \binom{2m}{2k+1} - \binom{2m}{2k} = 0$. [$\binom{2m}{2m+1} = 0$.]

7 I roll 3 fair dice. Which of the following sets of events are independant?

(a) (i) The first die is 4 (ii) The second die is 3 (iii) The total is even.

(b) (i) The total is 10 (ii) The first die is 3

(c) (i) The total of the first two dice is 7 (ii) The total of the second and third dice is 7

(d) (i) The total of the first two dice is 7 (ii) The total of the second and third dice is 7 (iii) The total of the first and third dice is 7

8 A fair die is rolled n times. Let p_n be the probability that no two consecutive rolls total 9.

(a) Show that the values p_n satisfy the recurrence $p_n = \frac{5}{6}p_{n-1} + \frac{1}{18}p_{n-2}$. [Hint: divide into two cases – sequences in which the second last roll is a 1 or 2, and sequences in which it is a 3, 4, 5, or 6.]

(b) Solve it to get a general formula for p_n.

(c) What is the expected number of rolls before two consecutive rolls total 9?

9 Show that if we have 6 2-element subsets of $\{1, 2, 3, 4, 5, 6\}$, then some two of them must be disjoint. [Hint: any two of the sets $\{1, 2\}$, $\{3, 4\}$, and $\{5, 6\}$ are disjoint, for example.]