MATH 2113/CSCI 2113, Discrete Structures II
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Mock Midterm Examination

Calculators not permitted. Answers may be left in reasonably simplified forms – e.g. binomial coefficients, factorials, etc.

These questions should probably take more than an hour and a half. This is so that I can include a larger variety of questions, to give a better idea of the sorts of questions that might appear on the actual midterm.

Compulsory questions

1 *Each course is graded by assigning one of the 11 grades A+, A, A-, B+, B, B-, C+, C, C-, D, or F to each student in the course. Suppose a given course has 15 students.*

(a) In how many ways can grades be assigned to students in the course.

There are 11 choices of grade for each student, and there are 15 students, so there are a total of 11^{15} ways to assign grades.

(b) Suppose the instructor also has to produce a summary stating how many of each grade have been given out, but not to which students. How many different summaries are possible?

Now consider the grades as boxes, and we wish to assign 15 items between the boxes. We can represent this by putting an 'x' for each student, and a vertical line for the boundary between two grades. The line then consists of a total of 15 crosses and 10 vertical lines. These can be arranged in $\binom{25}{10}$ ways. Therefore the number of possible summaries is $\binom{25}{10}$.

2 *Recall that a derangement of n elements is a permutation with no fixed points – i.e. $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that there are no values of m for which $\sigma(m) = m$. Use the inclusion-exclusion principle to find the number of derangements of an n-element set. [You can leave your answer expressed as a sum. Hint: use inclusion-exclusion to find the number of permutations with at least one fixed point.]*

Let A_i be the set of permutations that fix i. We then want to find the size of the complement of the union of all the A_i. Note that each A_i has $(n-1)!$ members – one point is fixed, and the other points can be arranged in any way whatsoever. Similarly, the intersection of any k of the A_i has $(n-k)!$ members. By the inclusion-exclusion principle, the union therefore has

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)! = -\sum_{k=1}^{n}(-1)^k\frac{n!}{k!}$$

members. Therefore the number of derangements is

$$n! + \sum_{k=1}^{n} (-1)^k \frac{n!}{k!} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

3 *Define a recurrence by* $a_n = 3a_{n-1} - 2a_{n-2} - 13n, a_0 = 0, a_1 = 4$.

(a) Find the generating function for the sequence a_n.

Multiply the nth equation by x^n and add to get

$$\sum_{n=2}^{\infty} a_n x^n = 3 \sum_{n=2}^{\infty} a_{n-1} x^n - 2 \sum_{n=2}^{\infty} a_{n-2} x^n - \sum_{n=2}^{\infty} 13 n x^n$$

Now we let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and the previous equation becomes

$$f(x) - a_1 x - a_0 = 3x(f(x) - a_0) - 2x^2 f(x) - \left(\frac{13x}{(1-x)^2} - 13x \right)$$

which we rearrange to get $f(x)(1 - 3x + 2x^2) = 4x + 13x - \frac{13x}{(1-x)^2} = x\frac{17(1-x)^2 - 13}{(1-x)^2} = x\frac{4 - 34x + 17x^2}{(1-x)^2}$. Therefore,

$$f(x) = \frac{x(17x^2 - 34x + 4)}{(1-x)^2(1 - 3x + 2x^2)} = \frac{x(17x^2 - 34x + 4)}{(1-x)^3(1-2x)}$$

(b) Find a general formula for a_n.

We can expand $f(x)$ as a partial fraction. We let

$$f(x) = \frac{x(17x^2 - 34x + 4)}{(1-x)^3(1-2x)} = \frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} + \frac{C}{1-x} + \frac{D}{1-2x}$$

We multiply through to get

$$A(1-2x) + B(1-x)(1-2x) + C(1-x)^2(1-2x) + D(1-x)^3 = x(17x^2 - 34x + 4)$$

Substituting $x = \frac{1}{2}$ and $x = 1$ gives $D = -35$ and $A = 13$ respectively. Setting $x = 0$ gives $B + C = 22$, and setting $x = 2$ gives $-39 + 3B - 3C + 35 = 8$ or $B - C = 4$, so $B = 13$ and $C = 9$.

This gives $f(x) = \frac{13}{(1-x)^3} + \frac{13}{(1-x)^2} + \frac{9}{1-x} - \frac{35}{1-2x}$, so that

$$a_n = 13\binom{n+2}{2} + 13\binom{n+1}{n} + 9 - 35 \times 2^n = \frac{13}{2}n^2 + \frac{65}{2}n + 35 - 35 \times 2^n$$

4 *(a) How many sets of 5 distinct cards from a standard deck contain an even number of kings?*

The number of kings is between 0 and 4. Of these possibilities, 0, 2, and 4 are even. There are 48 hands with 4 kings – just pick the one other card.

There are $\binom{4}{2}\binom{48}{3}$ hands with two kings, and there are $\binom{48}{5}$ hands with no kings. Therefore, the total number of hands with an even number of kings is $48+\binom{4}{2}\binom{48}{3}+\binom{48}{5}$.

(b) What is the probability that a 5 card poker hand is a full house given that it contains an even number of kings? [Recall that a full house is a hand with 3 cards of one rank and two cards of another.]

A hand with 4 kings cannot be a full house. There are $12\binom{4}{3}\binom{4}{2}$ full houses with 2 kings (choose the rank of the triple, and choose the suits). There are $12\times 11\binom{4}{3}\binom{4}{2}$ full houses with no kings (choose the ranks then choose the suits). Therefore, the probability that a hand is a full-house given that it contains an even number of kings is

$$\frac{12\times 12\binom{4}{3}\binom{4}{2}}{48+\binom{4}{2}\binom{48}{3}+\binom{48}{5}}$$

(c) What is the probability that a 5-card poker hand contains an even number of kings given that it is a full house?

The number of full houses with an even number of kings is $12\times 12\times 4\times 6$. The total number of full houses is $13\times 12\times 4\times 6$, So the probability that it contains an even number of kings is $\frac{12}{13}$.

[Another way to look at this is that the triple is equally likely to be any of the 13 ranks, so the probability that it is not kings is $\frac{12}{13}$. In this case, the hand has an even number of kings.]

5 *(a) A five-card hand is dealt at random from a standard 52 card deck. What is the probability that it contains the king of hearts.*

The probability that it contains the king of hearts is $\frac{5}{52}$ – we can consider it as the disjoint union of the events A_i that the ith card is the king of hearts. These are mutually exclusive, so the probability that one occurs is just the sum of their probabilities, or $\frac{5}{52}$.

(b) A five-card hand is dealt at random from each of 5 standard decks (so 5 hands are dealt in total). What is the probability that one of the hands contains the king of hearts?

For each hand, the probability that it does not contain the king of hearts is $\frac{47}{52}$, so the probability that none of the hands contains the king of hearts is $\left(\frac{47}{52}\right)^5$. Therefore, the probability that one of them contains the king of hearts is $1-\left(\frac{47}{52}\right)^5$.

6 *Show that $\sum_{k=0}^{m}\binom{2m}{2k+1}-\binom{2m}{2k}=0$. [$\binom{2m}{2m+1}=0$.]*

We can rewrite this sum as $\sum_{i=0}^{2m+1}(-1)^{i+1}\binom{2m}{i}=-\sum_{i=0}^{2m}(-1)^{i}\binom{2m}{i}$ By the binomial theorem, this is $(1+(-1))^{2m}=0^{2m}=0$.

7 *I roll 3 fair dice. Which of the following sets of events are independant?*

(a) (i) The first die is 4 (ii) The second die is 3 (iii) The total is even.

The probability that the first die is 4 is $\frac{1}{6}$. The probability that the second die is 3 is $\frac{1}{6}$. The probability that the total is even is $\frac{1}{2}$. (Whether the total of the first two dice is even or odd, there is a $\frac{1}{2}$ chance that the third die will be the same, making the total even.)

The probability that the first die is 4 and the second die is 3 is $\frac{1}{36}$, so these are independant. The probability that the first die is 4 and the total is even is $\frac{1}{12}$, since the total of the second and third die has a $\frac{1}{2}$ probability of being even. Similarly, the probability that the second die is 3 and the total is even is $\frac{1}{12}$. Finally, the probability of all 3 events is $\frac{1}{72}$ since there are 3 rolls that satisfy all 3.

$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$, $\frac{1}{12} = \frac{1}{6} \times \frac{1}{2}$, and $\frac{1}{72} = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{2}$. Therefore, the events are independant.

(b) (i) The total is 10 (ii) The first die is 3

The probability of both events is $\frac{1}{36}$ – there are 6 rolls that satisfy both conditions. The probability of the second is $\frac{1}{6}$. The probability of the first is $\frac{1}{6}\left(\frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36}\right) = \frac{1}{6} \times \frac{27}{36} = \frac{1}{8}$. Therefore, the events are not independant.

(c) (i) The total of the first two dice is 7 (ii) The total of the second and third dice is 7

The probability of (i) is $\frac{1}{6}$. The probability of (ii) is $\frac{1}{6}$. The probability of both events is $\frac{1}{36}$. (For any value n of the second die, the first and third must both be $7-n$, so there are 6 possibilities.) Therefore, the events are independant.

(d) (i) The total of the first two dice is 7 (ii) The total of the second and third dice is 7 (iii) The total of the first and third dice is 7

The probability of all 3 events is 0 – all 3 dice would have to be the same, and would therefore have to be half of 7, which is not a possible roll. Therefore, the events are not independant,

[But they are pairwise independant.]

8 *A fair die is rolled n times. Let p_n be the probability that no two consecutive rolls total 9.*

(a) Show that the values p_n satisfy the recurrence $p_n = \frac{5}{6}p_{n-1} + \frac{1}{18}p_{n-2}$. [Hint: divide into two cases – sequences in which the second last roll is a 1 or 2, and sequences in which it is a 3, 4, 5, or 6.]

Given a valid sequence of length $n-2$, there is a $\frac{1}{3}$ probability of extending it to a valid sequence of length n whose second last roll is 1 or 2 – once we roll a 1 or 2, the last roll is certain to be valid.

On the other hand, given a valid sequence of length $n-1$, the probability of extending it to a valid sequence of length n is $\frac{5}{6}$ if its last digit is 3, 4, 5, or 6, and 1 if the last roll is 1 or 2. The probability of a valid sequence of length $n-1$ ending in a 1 or 2 is $\frac{1}{3}p_{n-2}$. Therefore,

$$p_n = \frac{5}{6}\left(p_{n-1} - \frac{1}{3}p_{n-2}\right) + \frac{1}{3}p_{n-2} = \frac{5}{6}p_{n-1} + \frac{1}{18}p_{n-2}$$

(b) Solve it to get a general formula for p_n.

This is a second-order homogeneous constant-coefficient linear recurrence, so we know the general solutions are of the form $p_n = t^n$ for some t. Substituting, we get $t^2 = \frac{5}{6}t + \frac{1}{18}$, or $t = \frac{\frac{5}{6} \pm \sqrt{\frac{25}{36} + \frac{2}{9}}}{2} = \frac{5 \pm \sqrt{33}}{12}$

We know that $p_0 = 1$ and $p_1 = 1$. The solution is of the form $p_n = A\left(\frac{5+\sqrt{33}}{12}\right)^n + B\left(\frac{5-\sqrt{33}}{12}\right)^n$. From the values we know, we get $A + B = 1$ and $A(5+\sqrt{33}) + B(5-\sqrt{33}) = 12$ to give $2A\sqrt{33} = 12 - (5-\sqrt{33})$ or $A = \frac{7+\sqrt{33}}{2\sqrt{33}}$ and $B = -\frac{7-\sqrt{33}}{2\sqrt{33}}$. The solution is then

$$p_n = \frac{7+\sqrt{33}}{2\sqrt{33}}\left(\frac{5+\sqrt{33}}{12}\right)^n - \frac{7-\sqrt{33}}{2\sqrt{33}}\left(\frac{5-\sqrt{33}}{12}\right)^n$$

(c) What is the expected number of rolls before two consecutive rolls total 9?

Let N be the number of rolls until two consecutive rolls total 9. We know that $\mathbb{E}(N) = \sum_{n=1}^{\infty} n\mathbb{P}(N=n) = \sum_{n=0}^{\infty}\sum_{i=1}^{n}\mathbb{P}(N=n) = \sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\mathbb{P}(N=n) = \sum_{i=1}^{\infty}\mathbb{P}(N \geqslant i) = \sum_{j=0}^{\infty}\mathbb{P}(N > i) = \sum_{j=0}^{\infty} p_n$

We substitute to get

$$\begin{aligned}\mathbb{E}(N) &= \sum_{i=0}^{\infty}\frac{7+\sqrt{33}}{2\sqrt{33}}\left(\frac{5+\sqrt{33}}{12}\right)^i - \sum_{i=1}^{\infty}\frac{7-\sqrt{33}}{2\sqrt{33}}\left(\frac{5-\sqrt{33}}{12}\right)^i \\ &= \frac{7+\sqrt{33}}{2\sqrt{33}}\frac{1}{1-\frac{5+\sqrt{33}}{12}} - \frac{7-\sqrt{33}}{2\sqrt{33}}\frac{1}{1-\frac{5-\sqrt{33}}{12}} \\ &= \frac{6(7+\sqrt{33})}{\sqrt{33}(7-\sqrt{33})} - \frac{6(7-\sqrt{33})}{\sqrt{33}(7+\sqrt{33})} = \frac{6(164)}{16\sqrt{33}} = \frac{3}{2}\sqrt{33}\end{aligned}$$

9 *Show that if we have 6 2-element subsets of $\{1,2,3,4,5,6\}$, then some two of them must be disjoint. [Hint: any two of the sets $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$ are disjoint, for example.]*

We can partition the 15 2-element subsets of $\{1,2,3,4,5,6\}$ into 5 pairwise disjoint triples, for example:

$$\{1,2\} \quad \{3,4\} \quad \{5,6\}$$

$$\begin{array}{ccc} \{1,3\} & \{2,5\} & \{4,6\} \\ \{1,4\} & \{2,6\} & \{3,5\} \\ \{1,5\} & \{2,4\} & \{3,6\} \\ \{1,6\} & \{2,3\} & \{4,5\} \end{array}$$

If we have 6 subsets with two elements, then by the pigeon-hole principle, some two must come from the same triple, so they must be disjoint.