MATH 2600/STAT 2600, Theory of Interest
FALL 2013
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Midterm Examination
Model Solutions

1. *Mr and Mrs. Allen are saving up for their children's education. They have two children, aged 9 and 11. They invest \$40,000 today at $j_1 = 8\%$ interest, and they want to divide this equally among their two children: when each child is 18 (at exactly this time of year), they will receive their share X. How much does each child get?*

   We set up an equation of value with focal date in 9 years time. This gives $40000(1.08)^9 = X + (1.08)^2X$, so $X = \frac{40000(1.08)^9}{1+1.08^2} = 36909.24$.

2. *Which of the following interest rates is best for the borrower?*

   (i) 8% compounded quarterly

   (ii) 8.1% compounded annually

   (iii) 8.2% compounded monthly

   8.2% compounded monthly is clearly worse for the borrower than the other two. 8% compounded quarterly is an annual effective rate of $(1.02)^4 - 1 = 0.0824$, so (ii) is best for the borrower.

3. *If force of interest is given by $\delta_t = 0.15 + 0.1t - 0.15t^2$ over a one-year period, what is the accumulated value, at the end of the year, of \$6,000 invested at the start of the year?*

   The accumulated value is given by $A = 6000e^{\int_0^1 \delta_t dt}$. We have $\int_0^1 0.15 + 0.1t - 0.15t^2 dt = [0.15t + 0.05t^2 - 0.05t^3]_0^1 = 0.15$, so the accumulated value is $6000e^{0.15} = 6971.01$.

4. *The stock of company ABC currently pays a dividend of \$0.45 every quarter. Every quarter the company increases the dividend by 0.5%. The current price for the stock (just after a dividend of \$0.45 is payed) is \$20. What interest rate is being used to value this stock?*

   The last dividend was \$0.45. If this dividend were constant, then for a price of \$20, the rate of interest would be 2.25% per quarter (or $j_4 = 9\%$). However, this is the "real" rate of interest $\frac{i-r}{1+r}$, where $r = 0.5\%$ per quarter, so we have $i = 2.25(1.005) + 0.5 = 2.76\%$ per quarter. The interest rate being used is therefore $j_4 = 11.04\%$.

5. *A company buys a machine for \$32,000. The machine is expected to last for 5 years, after which it will have a salvage value of \$1,000. Prepare a depreciation schedule using:*

   (a) The constant percentage method

We have $32000(1-d)^5 = 1000$, so $1-d = 0.5$, and $d = 0.5$.

Year	Value at Start of year	depreciation
1	32,000	16,000
2	16,000	8,000
3	8,000	4,000
4	4,000	2,000
5	2,000	1,000

(b) The straight line method

The depreciation each year is $\frac{31000}{5} = 6200$.

Year	Value at Start of year	depreciation
1	32,000	6,200
2	25,800	6,200
3	19,600	6,200
4	13,400	6,200
5	7,200	6,200

6. *A company are deciding between two machines. The first machine costs \$100,000, and lasts for 7 years, after which it has a resale value of \$6,000, and has maintainance costs of \$5,000 every year. The second machine costs \$200,000, lasts for 15 years, with a resale value of \$25,000, and has maintainance costs of \$2,000 every year. If the cost of capital is $j_1 = 6\%$, which machine has lower total capitalised cost?*

   The replacement cost for the first machine is \$94,000 every 7 years, which has a present value of $\frac{94000}{1.06^7-1} = 186644.86$. The maintainance cost is a perpetuity of \$5,000 annually, which has present value $\frac{5000}{0.06} = 83333.33$, so the total recapitalised cost is $100000+186644.86+83333.33 = \$369,978.19$.

   The replacement cost for the second machine is \$175,000 every 9 years, which has a present value of $\frac{175000}{1.06^{15}-1} = 125308.06$. The maintainance cost is a perpetuity of \$2,000 annually, which has present value $\frac{2000}{0.06} = 33333.33$, so the total recapitalised cost is $200000+125308.06+33333.33 = \$358,641.39$.

   Therefore, the second machine has lower total capitalised cost.

7. *A loan of \$100,000 at $j_1 = 6\%$ is amortised with equal annual payments for 4 years.*

   (a) Calculate the annual payments.

   The annual payments are given by $Ra_{\overline{4}|0.06} = 100000$, so $R = \frac{100000\times0.06}{1-1.06^{-4}} = 28859.15$.

   (b) Draw up a complete amortisation schedule for the loan.

Year	Outstanding Balance	Interest	Payment	Principal Repaid
1	100000.00	6000	28859.15	22859.15
2	77140.85	4628.45	28859.15	24230.70
3	52910.15	3174.61	28859.15	25684.54
4	27225.61	1633.54	28859.15	27225.61

8. *Mr. Brooks buys a house in the US. He needs to borrow \$200,000 at $j_{12} = 6\%$, amortised over 20 years. There is also a financing fee of \$3,000. What is the APR for this loan?*

   The loan is for \$203,000, amortised over 20 years at $j_{12} = 6\%$, so the monthly payments are given by $Ra_{\overline{240}|0.005} = 203000$, so $R = \frac{203000\times 0.005}{1-1.005^{-240}} = 1454.36$. We want to find the APR, or the rate such that these repayments would repay a loan of \$200,000. That is, we need to solve $1454.36a_{\overline{240}|i} = 200000$ or $a_{\overline{240}|i} = \frac{200000}{1454.36} = 137.518$.

i	$a_{\overline{240}\|i}$
0.0050	139.581
0.0055	133.072
0.0052	136.922
0.00515	137.579
0.005151	137.566
0.005153	137.540
0.005160	137.447
0.005155	137.513

   Therefore the monthly rate is 0.5155%, so the APR is $1.005155^{12} - 1 = 6.36\%$.

9. *Mrs. Carle borrows \$6,000 for one year at 6% simple interest. After 2 months, she repays \$3,000.*

   (a) If the loan is calculated using the merchant's rule, how much does she need to pay at the end of the year, to pay off the debt?

   Using the merchant's rule, the loan increases to $6000(1.06) = 6360$ at the end of the year. The first repayment increases to $3000(1+\frac{10}{12}0.06) = 3150$, so the balance due is $6360 - 3150 = 3210$.

   (b) What if the loan is calculated using the US rule?

   Using the US rule, after 2 months, the balance is $6000(1+\frac{2}{12}0.06) = 6060$. After the repayment, the balance becomes 3060, so at the end of the year, the balance is $3060(1+\frac{10}{12}0.06) = 3213$.