MATH 2600/STAT 2600, Theory of Interest
FALL 2013
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Homework Sheet 1
Model Solutions

1. *Calculate the accumulated value on maturity of the following investments:*

   (a) \$7,000 invested for 20 years at 2% effective annual interest.

   The accumulated value is $7000(1.02)^{20} = 10401.63$.

   (b) \$2,000 invested for 7 months at 4% simple interest.

   The accumulated value is $2000\left(1 + 0.04\frac{7}{12}\right) = 2093.33$.

   (c) \$3,000 invested for 4 years at $j_4 = 6\%$.

   The accumulated value is $3000(1.015)^{16} = 3806.96$.

2. *Mr. Brown invests \$1,000 at 8% effective annual interest.*

   (a) How many years does he have to wait before he has saved up \$1,800?

   After t years, he has $1000(1.08)^t$, so we solve

   $$1.08^t = \frac{1800}{1000}$$

   to get

   $$t = \frac{\log(1.8)}{\log(1.08)} = 7.637$$

   years.

   (b) If he needs his investment to grow to \$1,800 within 8 years, what rate of interest does he need to invest at?

   In this case, he needs

   $$1000(1+i)^8 = 1800$$

   , which gives

   $$1 + i = \left(\frac{1800}{1000}\right)^{\frac{1}{8}} = 1.0762$$

   , so he needs to invest at an effective annual rate of 7.62%.

3. *Mrs. Collins takes out a loan for \$15,000 at 6% effective annual rate. After 4 years, the interest rate drops to 5%. She repays \$7,000 after 6 years, and \$2,000 after 7 years. What is the outstanding balance after 8 years?*

   After 4 years, the balance is $15000(1.06)^4 = 18937.15$. After 6 years, the balance is $18937.15(1.05)^2 = 20878.21$ before her payment, so after her payment, it is \$13,878.21. After 7 years, this has grown to $13878.21 \times 1.05 = 14572.12$, and after her payment, the balance is \$12,572.12. After the eighth year, the balance is therefore $12572.12 \times 1.05 = 13200.73$.

4. *Mr. Donaldson receives an invoice for $1,200, for payment within 40 days. He can get a 3% discount if he pays within the first 10 days. What is the largest rate of simple interest at which it would be worth his taking out a loan to get the discount.*

   If Mr. Donaldson takes out the loan, he will pay $1200 \times 0.97 = 1164$ after 10 days. Then 30 days later, he will have to pay $1164\left(1+\frac{30}{365}i\right)$ to settle the loan. If he does not take out the loan, then he needs to pay $1,200 after 40 days. The two options are the same when $1164\left(1+\frac{30}{365}i\right) = 1200$, or when $1+\frac{30}{365}i = \frac{1200}{1164}$, which gives $\frac{30}{365}i = \frac{1200}{1164}-1$, or $i = \frac{365\times 36}{30\times 1164} = 37.6\%$. So it is worth taking out the loan at any rate less than 37.6%.

5. *Mrs. Erikson buys a promissory note for $8,000 in 200 days at 4% simple interest. After 60 days, she sells it to a bank, which discounts notes at 4% simple interest.*

   (a) How much does the bank pay for the note?

   If the bank pays P, then we have $P\left(1+\frac{140}{365}0.04\right) = 8000\left(1+\frac{200}{365}0.04\right)$ or $P = 8051.81$.

   (b) What is Mrs. Erikson's rate of return?

   Mrs. Erikson pays $8000 for the note, and receives $8051.81 after 60 days. After 60 days, she has achieved $\frac{8051.81-8000}{8000} = 0.00648$ times her investment. The annual rate of return is then given by $\frac{60}{365}i = 0.00648$, which gives $i = \frac{365}{60} \times 0.00648$, so her annual rate of simple interest return is 3.94%.

6. *Mr. Fox wants to save up $800,000 for his retirement in 14 years time.*

   (a) If his effective annual rate of interest is 6%, how much does he need to invest now?

   We need to solve $(1.06)^{14}P = 800000$, so $P = 800000(1.06)^{-14} = 353840.77$.

   (b) Suppose he invests this much, but the actual rate of interest he receives is 7%. How much earlier can he retire?

   If the actual rate of return is 7%, then he can retire when $353840.77(1.07)^t = 800000$, or $(1.07)^t = 2.261$. The solution is $t = \frac{\log 2.261}{\log 1.07} = 12.06$ years, so he can retire almost 2 years earlier.

7. *What rate of simple discount is equivalent to 9% simple interest over a period of 7 months?*

   To get the value after 7 months, we multiply by $1+\frac{7}{12}0.09 = 1.0525$. To get from the value in 7 months to the current value, we multiply by $\frac{1}{1.0525} = 0.95012$. The discount for 7 months is therefore 4.988%, so the annual rate of discount is $\frac{12}{7}0.04998 = 0.0855$, so 8.55%.

8. *What price should you pay for a T-bill of face value $5,000, maturing in 74 days at a simple interest rate of 4%?*

   The price should satisfy $P\left(1+\frac{74}{365}0.04\right) = 5000$, which gives $P = 4959.78$.

9. *Mr and Mrs. Graham are saving up for their children's education. They have two children, aged 8 and 12. They invest \$80,000 today at $j_{12} = 4\%$ interest, and want to use this to pay \$50,000 for each child's education, when the child is 18 (at exactly this time of year). How much money can they take out from this fund in two years time, and still have enough?*

   We use an equation of value. For convenience, we choose focal date in two years time. The value of their investment at this time is $80000(1.003333333)^{24}$. The present values of the payments they need for their children's education are $50000(1.003333333)^{-48}$ and $50000(1.0033333333)^{-96}$. The equation of value is therefore:

   $$80000(1.003333333)^{24} = X + 50000(1.00333333)^{-48} + 50000(1.0033333)^{-96}$$

   which gives $X = 7706.13$.

10. *Which of the following interest rates is best for the borrower?*

    (i) 8% compounded quarterly

    (ii) 8.2% compounded annually

    (iii) 7.8% compounded daily

    The annual effective rates are:

    (i) $1.02^4 - 1 = 8.243\%$

    (ii) 8.2%

    (iii) $\left(1 + \frac{0.078}{365}\right)^{365} - 1 = 8.11\%$

    The lowest rate is best for the borrower, so (iii) is best.

11. *What rate of compound interest is equivalent to an 8% rate of compound discount?*

    This is given by $\frac{1}{0.92} - 1 = 8.70\%$.

12. *What rate of continuous compounding (constant force of interest) is equivalent to annual effective rate of 4%?*

    We have that $e^{\delta_t} = 1.04$, so $\delta_t = \log(1.04) = 0.03922$.

13. *If force of interest is given by $\delta_t = 0.1 - 0.3t + 0.1t^2$ over a two-year period, what is the accumulated value at the end of that period, of \$5,000 invested at the start of the period?*

    The accumulated value is given as $5000e^{\int_0^1 \delta_t dt}$. The integral is $\int_0^1 (0.1 - 0.3t + 0.1t^2)dt = [0.1t - 0.15t^2 + 0.03333333t^3]_0^1 = -0.0166666667$. The accumulated value is therefore \$4917.36.

14. *Mr. Harris wants to buy a house. Today, he would need a downpayment of \$150,000 to buy the house, but he only has \$90,000. He invests this money at $j_{12} = 7\%$. However, the downpayment needed increases with*

inflation at a rate of $j_{12} = 6\%$*. How long must he wait before he has saved up enough to make the downpayment?*

The real annual effective rate of return is $\frac{\left(1+\frac{0.07}{12}\right)^{12}}{\left(1+\frac{0.06}{12}\right)^{12}} - 1 = 0.9996\%$. He needs his money to increase by a factor of $\frac{150000}{90000}$. The time this takes is given by $\frac{\log 1.6666666666666667}{\log 1.009996} = 51.36$ years.