MATH 2600/STAT 2600, Theory of Interest
FALL 2013
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Homework Sheet 2
Model Solutions

1. *Mr. Irving invests \$300 a month at $j_{12} = 5\%$ into a fund for his granddaughter's education. How much is in the fund when she starts university 8 years 4 months from the first investment in the fund?*

   The amount in the fund is given by $300s_{\overline{101}|\frac{0.05}{12}} = 300\frac{\left(1+\frac{0.05}{12}\right)^{101}-1}{\frac{0.05}{12}} = 37576.4$.

2. *Mrs. Jones wants to save up \$600,000 for her retirement. She starts a savings account, which pays interest at $j_{12} = 4\%$. She plans to make monthly contributions into the account from now until her retirement in 22 years 4 months time.*

   (a) How much should she pay into the account each month in order to meet her retirement goal?

   If her monthly payments are R, then the amount she saves up is $Rs_{\overline{269}|0.003333333}$, so we need to solve $Rs_{\overline{269}|0.003333333} = 600000$, or $R\frac{(1.00333333)^{269}-1}{0.003333333)} = 600000$, which gives $R = 1381.44$.

   (b) If she can only afford \$900 a month, how long does she have to delay her retirement?

   If she only pays \$900 a month, then after n months, she accumulates $900s_{\overline{n}|0.0033333333} = 900\frac{(1.00333333333)^n-1}{0.00333333}$. We need to solve $900\frac{(1.00333333333)^n-1}{0.00333333} = 600000$, which gives $(1.0033333333)^n = \frac{0.003333333\times600000}{900} + 1$ or $n = \frac{\log 3.22222222222}{\log 1.0033333333} = 351.61$. She therefore has to wait $352 - 1 = 351$ months to retire. This is a delay of 83 months or just under 7 years.

   (c) If she does not want to delay her retirement, what interest rate does she need to receive on her savings?

   If she is paying \$900 a month, she needs $s_{\overline{n}|i} = \frac{600000}{900}$.

   We calculate for a few values of i in the following table, using linear interpolation:

i	$900s_{\overline{269}\|i}$
0.003333333	390896.66
0.005833333	583323.25
0.006666667	671461.99
0.005991009	598901.47
0.006001238	600000.07

   So the rate of return she needs is $j_{12} = 12 \times 0.006001238 = 7.20\%$.

3. *Mr. King borrows \$20,000 from a bank at $j_{12} = 7\%$. He plans to pay this off with monthly payments over the next 3 years.*

   (a) What are the monthly payments?

   The monthly payments are given by

$$\frac{20000}{a_{\overline{36}|0.0058333}} = \frac{20000 \times 0.005833333}{1 - 1.005833333^{-36}} = 617.542$$

   (b) After 1 year, the bank sells the loan to another bank, which wants to receive $j_{12} = 6.8\%$. How much does that bank pay for the loan?

   After 1 year, there are 24 payments remaining, so the value of these is $617.542a_{\overline{24}|0.005666667} = 617.542\frac{1-1.00566667^{-24}}{0.005666667} = 13820.83$.

4. *Dr. Lee is investing for her retirement. She makes monthly payments of \$300 into an account that pays $j_{12} = 6\%$, starting in January 2002. In January 2006, interest rates drop to $j_{12} = 4\%$. From August 2007, she increases her monthly payments to \$400. How much is in the account when she makes her last payment in March 2011?*

   After the first 4 years, the balance is $300s_{\overline{49}|0.005} = 300\frac{1.005^{49}-1}{0.005} = 16610.50$. This accumulates to $16610.50(1.003333333)^{62} = 20416.79$ by March 2011. The \$300 payments from February 2006 to March 2011 accumulate to $300s_{\overline{62}|0.003333333} = 300\frac{1.003333333333^{62}-1}{0.00333333333333} = 20623.51$, and the additional \$100 a month from August 2007 accumulates to $100s_{\overline{44}|0.003333333333} = 100\frac{1.003333333^{44}-1}{0.003333333333} = 4730.57$. The total in the account is therefore \$45,770.87.

5. *Miss MacDonald donates \$4,000,000 to her old university. She states that the donation should be used to fund an annual scholarship of \$8,000 for each of 15 students. At what interest rate does the money need to be invested to provide this scholarship forever.*

   At annual interest rate i, the money can fund annual payments of $4000000i$. We want this to equal \$120,000, so we get $i = \frac{120000}{4000000} = 0.03$.

6. *Mr. Neil deposits \$800 every quarter into an account which pays interest at $j_{12} = 9\%$. How much is in the account when he makes the 11th deposit?*

   The quarterly interest rate is $1.0075^3 - 1$, so the amount in the account at the time of the 11th deposit is

$$800s_{\overline{11}|1.0075^3-1} = 800\frac{1.0075^{33} - 1}{1.0075^3 - 1} = 9868.45$$

7. *Mrs. O'Riley is saving up to go on holiday. Every day she puts \$10 into an account which pays interest at $j_1 = 5\%$. How long does she have to wait before she has saved up \$1,500 for her holiday?*

After n payments, the amount in the account is

$$10s_{\overline{n}|1.05^{\frac{1}{365}}-1} = 10\frac{1.05^{\frac{n}{365}}-1}{1.05^{\frac{1}{365}}-1}$$

We want this to be 1500, so we need to solve

$$\begin{aligned}
10\frac{1.05^{\frac{n}{365}}-1}{1.05^{\frac{1}{365}}-1} &= 1500\\
1.05^{\frac{n}{365}}-1 &= 150(1.05^{\frac{1}{365}}-1)\\
n &= 365\frac{\log(1+150(1.05^{\frac{1}{365}}-1))}{\log 1.05} = 148.52
\end{aligned}$$

So she has to wait for the 149th payment, 148 days after her first payment.

8. *Mr. and Mrs. Purcell are retiring. They have saved up \$600,000, from which they want to live for the next 25 years. They want to take out monthly withdrawls, which will increase every month in line with inflation at an annual rate of 3%. (That is, the withdrawls form a geometric progression, with each payment 3% more than the one twelve months earlier.) If their money is invested at $j_{12} = 6\%$, how much should the first withdrawl be?*

   The monthly increase of withdrawls is by a factor $1.03^{\frac{1}{12}}$, so the real monthly rate of interest is $j = \frac{1.005-1.03^{\frac{1}{12}}}{1.03^{\frac{1}{12}}}$. They want to withdraw money for 25 years, so the present value of withdrawls, starting with $1.03^{\frac{1}{12}}R$ is $Ra_{\overline{300}|j} = R\frac{1-(1+j)^{-300}}{j} = 600000$, so $R = \$2853.18$. The first deposit is therefore \$2860.22.

9. *Mr. Quiggly takes out a loan for \$8,000 at $j_{12} = 7\%$. He wants to repay the loan with an increasing arithmetic progression of payments. He would like the first payment in one month's time to be \$100, and he would like the loan to be payed off after two years. By how much should the payments increase each month?*

   If the increment is R, then the value of the payments above \$100 is given by $R(\nu^2 + 2\nu^3 + \cdots + 23\nu^{24})$. Multiplying by $j+1-1$ (where $j = \frac{0.07}{12}$ gives $R(\nu^1 + \nu^2 + \cdots + \nu^{23} - 23\nu^{24})$, so we get

$$8000 = 100a_{\overline{24}|\frac{0.07}{12}} + R\frac{a_{\overline{23}|\frac{0.07}{12}} - 23\nu^{24}}{\frac{0.07}{12}} = 100\frac{12(1-(1+\frac{0.07}{12})^{-24})}{0.07} + R\frac{144(1-(1+\frac{0.07}{12})^{-23} - 12\times 0.07 \times}{0.07^2}$$

   We solve this to get $R = 23.01$.

10. *The stock of company XYZ currently pays a dividend of \$0.10 every month. Every year the company increases the dividend by 4%. What is a fair price for the stock at $j_{12} = 11\%$?*

The accumulated value of this year's dividends at the start of the year is $0.1s_{\overline{12}|\frac{0.11}{12}} = 0.1\frac{12(1-(1+\frac{0.11}{12})^{-12})}{0.11} = 1.131$. This annual perpetuity should then be valued at the "real" interest rate $\frac{(1+\frac{0.11}{12})^{12}-1.04}{(1+\frac{0.11}{12})^{12}} = 0.0679$, to get a present value of $\frac{1.131}{0.0679} = 16.67$.

11. *Mrs. Rogers makes a loan of $15,000 at* $j_{12} = 9\%$*. The loan is repaid over 4 years with equal monthly payments. When Mrs. Rogers receives each payment, she immediately deposits it in an account which receives* $j_{12} = 3\%$ *interest. What yield does she make on her investment at the end of the 4 years?*

    The monthly payments are given by $\frac{15000}{a_{\overline{48}|0.0075}} = \frac{15000\times0.0075}{1-1.0075^{-48}} = 373.28$.

    These accounts accumulate to $373.28s_{\overline{48}|0.0025} = 373.28\frac{1.0025^{48}-1}{0.0025} = 19011.38$.

    Her yield is therefore given by $\left(\frac{19011.38}{15000}\right)^{\frac{1}{4}} - 1 = 6.1\%$.

12. *A company buys a machine for $50,000. The machine is expected to last for 7 years, after which it will have a salvage value of $1,500. Prepare a depreciation schedule using:*

    (a) The sum of digits method.

Year	Value at start of year	Depreciation
0	50000	12125.00
1	37875.00	10392.86
2	27482.14	8660.71
3	18821.43	6928.57
4	11892.86	5196.43
5	6696.43	3464.29
6	3232.14	1732.14
7	1500	

(b) The constant percentage method

Year	Value at start of year	Depreciation
0	50000	19701.86
1	30298.14	11938.60
2	18359.54	7234.34
3	11125.20	4383.75
4	6741.45	2656.38
5	4085.07	1609.67
6	2475.40	975.40
7	1500	

(c) The straight line method

Year	Value at start of year	Depreciation
0	50000	6928.57
1	43071.43	6928.57
2	36142.86	6928.57
3	29214.29	6928.58
4	22285.71	6928.57
5	15357.14	6928.57
6	8428.57	6928.57
7	1500	

(d) The compound interest method, with cost of capital $j_1 = 6\%$

Sinking fund payments $\frac{48500}{s_{\overline{7}|0.06}} = \frac{48500\times 0.06}{1.06^7-1} = 5778.05$

Year	Value at start of year	Depreciation
0	50000	5778.05
1	44221.95	6124.73
2	38097.22	6492.22
3	31605.00	6881.75
4	24723.25	7294.65
5	17428.60	7732.33
6	9696.27	8196.27
7	1500	

13. *You are deciding between two cars. The first car costs \$16,000, lasts for 8 years, after which it has a resale value of \$1,600, and has fuel and maintainance costs of \$1,300 in the first year, and increasing by \$100 every subsequent year. The second car costs \$22,000, lasts for 9 years, with a resale value of \$3,200, and has fuel and maintainance costs of \$800 in the first year, and increasing by \$80 in each subsequent year.*

    (a) If the cost of capital is $j_1 = 8\%$, which car has lower total capitalised cost?

    For the first car, the accumulated value of the fuel and maintainance costs at the time the next new car is bought is an annuity of \$1,200, plus an increasing annuity, with first payment 100 and subsequent payments increasing by 100. The value of this is $100(1.08)^7 + 200(1.08)^6 + \cdots + 800 = 3295.78$. The value of the annuity of \$1,200 payments is $1200s_{\overline{8}|0.08} = 12763.95$. We add these costs and the costs of replacing the car, to get a perpetuity of \$30,459.74 every 8 years, at 8% interest per year. The present value of this perpetuity is given by $30,459.741.08^8 - 1 = \$35,795.81$. We add the original cost of the car, the get a total capitalised cost of \$51,795.81.

    For the second car, the accumulated value of the fuel and maintainance costs at the time the next new car is bought is an annuity of \$800, plus an increasing annuity, with first payment 80 and subsequent payments increasing by 100. The value of this is $80(1.08)^7 + 160(1.08)^6 + \cdots + 640 = 2636.63$. The value of the annuity of \$800 payments is $800s_{\overline{9}|0.08} =$

9990.05. We add these costs and the costs of replacing the car, to get a perpetuity of $31,278.60 every 9 years, at 8% interest per year. The present value of this perpetuity is given by $34,278.601.08^9 - 1 = \$31,309.76$. We add the original cost of the car, the get a total capitalised cost of $53,309.76. Therefore, the first car is cheaper.

(b) [bonus] At what cost of capital would the two cars have the same total capitalised cost?

We compute the total capitalised costs at various interest rates, to get

j_1	Cost of first car	Cost of second car	difference
8%	51795.81	53309.76	-1513.95
4%	93306.10	91889.86	1416.24
6%	65573.97	66084.74	-510.77
5%	76648.90	76380.73	268.17
5.3%	72884.08	72878.86	5.22
5.4%	71722.76	71799.00	-76.23
5.32%	72648.30	72659.60	-11.30
5.31%	72765.97	72769.02	-3.05
5.305%	72824.97	72823.89	1.08

So the cars have the same total capitalised cost at 5.31%.