MATH 2600/STAT 2600, Theory of Interest
FALL 2013
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Homework Sheet 3
Model Solutions

1. *A loan of \$15,000 at $j_{12} = 10\%$ is amortised with equal monthly payments for 2 years.*

   (a) Calculate the monthly payments.

   The monthly payments are given by $\frac{15000}{a_{\overline{24|}\frac{0.1}{12}}} = \frac{15000\times\frac{0.1}{12}}{1-\left(1+\frac{0.1}{12}\right)^{-24}} = 692.18$ (rounded up).

   (b) Draw up a complete amortisation schedule for the loan.

Month	Outstanding Balance	Payment	Interest	Principal Repaid
1	15000	692.18	125.00	567.18
2	14432.82	692.18	120.27	571.91
3	13860.91	692.18	115.51	576.67
4	13284.24	692.18	110.70	581.48
5	12702.76	692.18	105.86	586.32
6	12116.44	692.18	100.97	591.21
7	11525.23	692.18	96.04	596.14
8	10929.09	692.18	91.08	601.10
9	10327.99	692.18	86.07	606.11
10	9721.88	692.18	81.02	611.16
11	9110.71	692.18	75.92	616.26
12	8494.45	692.18	70.79	621.39
13	7873.06	692.18	65.61	626.57
14	7246.49	692.18	60.39	631.79
15	6614.70	692.18	55.12	637.06
16	5977.64	692.18	49.81	642.37
17	5335.27	692.18	44.46	647.72
18	4687.55	692.18	39.06	653.12
19	4034.44	692.18	33.88	658.56
20	3375.88	692.18	28.13	664.05
21	2711.83	692.18	22.60	669.58
22	2042.25	692.18	17.02	675.16
23	1367.09	692.18	11.39	680.79
24	686.30	692.02	5.72	686.30

2. *Mrs. Slater takes out a 20-year mortgage for a loan of \$400,000 at $j_2 = 6\%$.*

   (a) Calculate the monthly payments required.

The monthly payments are given by $\frac{400000}{a_{\overline{240}|1.03^{\frac{1}{6}}-1}} = \frac{15000(1.03^{\frac{1}{6}}-1)}{1-1.03^{-40}} = 2848.76$ (rounded up).

(b) After 5 years, the interest rate drops to $j_2 = 5\%$, calculate the new monthly payments if she wishes to keep the mortgage over 20 years.

After 5 years the outstanding balance is $400000(1.03)^{10} - 2848.76 s_{\overline{60}|1.03^{\frac{1}{6}}} = 400000(1.03)^{10} - 2848.76\frac{1.03^{10}-1}{1.03^{\frac{1}{6}}-1} = 339184.25$. The new payments are therefore given by $\frac{339184.25}{a_{\overline{180}|1.025^{\frac{1}{6}}-1}} = 339184.25\frac{1.025^{\frac{1}{6}}-1}{1.025^{30}-1} = 2673.20$ (rounded up).

(c) If instead, she wishes to keep the mortgage payments the same, when will she finish paying off the mortgage?

If the mortgage payments remain at \$2848.76, the present value of n future payments is $2848.76 a_{\overline{n}|1.025^{\frac{1}{6}}-1}$. We therefore need to solve $2848.76 a_{\overline{n}|1.025^{\frac{1}{6}}-1} = 339184.25$, so that

$$
\begin{aligned}
a_{\overline{n}|1.025^{\frac{1}{6}}-1} &= \frac{339184.25}{2848.76} \\
\frac{1-1.025^{-\frac{n}{6}}}{1.025^{\frac{1}{6}}-1} &= \frac{339184.25}{2848.76} \\
1-1.025^{-\frac{n}{6}} &= (1.025^{\frac{1}{6}}-1)\frac{339184.25}{2848.76} \\
1.025^{-\frac{n}{6}} &= 1-(1.025^{\frac{1}{6}}-1)\frac{339184.25}{2848.76} \\
n &= -6\frac{\log\left(1-(1.025^{\frac{1}{6}}-1)\frac{339184.25}{2848.76}\right)}{\log(1.025)} \\
n &= 164.10
\end{aligned}
$$

So the mortgage would have 164 months remaining, or 13 years 8 months. [So the total term of the mortgage would be 18 years 8 months.]

3. *Mr. and Mrs. Thomas buy a cottage, with a downpayment of \$50,000 and a 15-year mortgage for the remaining \$150,000 at $j_2 = 5\%$. There is a penalty of three times monthly interest on the outstanding balance for paying off the loan early. After 3 years, another company offers them a chance to refinance at $j_2 = 4.4\%$ for the remaining 12 years of the loan. Should they refinance?*

   The monthly payments are given by $\frac{150000}{a_{\overline{180}|1.025^{\frac{1}{6}}-1}} = \frac{150000(1.025^{\frac{1}{6}}-1)}{1-1.025^{-30}} = 1182.19$ (rounded up).

After 3 years, the outstanding balance is $150000\times1.025^6-1182.19s_{\overline{36}|1.025^{\frac{1}{6}}-1}=150000\times1.025^6-1182.19\frac{1.025^6-1}{1.025^{\frac{1}{6}}-1}=128175.37$. Three times the monthly interest on this is $3\times128175.37\times(1.025^{\frac{1}{6}}-1)=1585.75$, so if they refinance, the amount payable is \$129,761.12. The monthly payments are then $\frac{129761.12}{a_{\overline{144}|1.022^{\frac{1}{6}}-1}}=\frac{128175.36(1.022^{\frac{1}{6}}-1)}{1-1.022^{-24}}=1158.93$, so it would be cheaper for them to refinance.

4. *Mrs. Upson buys a house in the US. She needs to borrow \$400,000 at $j_{12}=6.4\%$, amortised over 20 years. There is also a financing fee of \$4,000. What is the APR for this loan?*

   The monthly repayments are given by $\frac{404000}{a_{\overline{240}|0.005333333333}}=\frac{404000\times0.0053333333333}{1-1.005333333333333^{-240}}=2988.38$. These are to repay a loan of \$400,000. We calculate present values at a few different interest rates.

Interest rate	Present Value
6.400%	404000.00
6.500%	400816.22
6.600%	397669.88
6.515%	400341.90
6.525%	400026.15
6.535%	399710.77

   So the APR is $j_{12}=6.53\%$, or an annual rate of 6.73%.

5. *Mrs. Valdez borrows \$300,000 to invest in the stock market. She has two options for repayment. She can either amortise the loan over 20 years at $j_{12}=3\%$, or she can make interest only payments at $j_{12}=3.4\%$ for 20 years, then pay off the balance with a lump sum payment at the end of the 20 years. What rate of return does she need on her investments to make the interest-only payments the better deal?*

   If she amortises the loan, then her monthly payments are $\frac{300000}{a_{\overline{240}|0.0025}}=\frac{300000\times0.0025}{1-1.0025^{-240}}=1663.79$. If she makes interest-only payments, then her monthly payments are $\frac{10200}{12}=850$, and she has the final payment of \$300,000 after 20 years, so for the interest-only payments to be better, she needs the accumulated value of the money saved on the monthly payments to be greater than the lump-sum repayment. That is, she needs $813.79s_{\overline{240}|i}>300000$, or $s_{\overline{240}|i}>368.644$. We determine the rate i by trying a few values:

i	$s_{\overline{240}\mid i}$
0.500%	462.04
0.300%	350.74
0.340%	370.09
0.335%	367.60
0.337%	368.59
0.3375%	368.84

So the monthly rate of return needed is 0.337%, which is $j_{12} = 4.04\%$.

6. *A bank lends $400,000 to Mr. and Mrs. Wilson. The loan is payed back with monthly interest-only payments at $j_{12} = 5\%$, with the principal returned as a lump sum after 25 years. After 8 years, the bank sells the loan to a private investor, who wishes to achieve an annual effective yield of 5.4%.*

   (a) How much does the investor pay for the loan?

   The remainder of the loan consists of 204 monthly payments of $1,666.67, and a final payment of $400,000 after 17 years. We can find the present value using Makeham's formula. The present value of the principal is $400000 \times 1.054^{-17} = 163594.84$, and the monthly interest rate is $1.054^{12} - 1 = .00439$. Therefore, Makeham's formula gives the present value of the loan as $163594.84 + (400000 - 163594.84)\frac{\frac{0.05}{12}}{0.00439} = 387854.69$.

   (b) If the bank wants to make an annual effective return of 4.8% on its investment, what annual effective yield would the buyer receive?

   If the bank wants an annual return of 4.8%, then the value of the original loan is 400000×1.048^8, and the value of the payments received is $1666.67 s_{\overline{96}\mid 1.048^{\frac{1}{12}}-1} = 1666.67 \frac{1.048^8 - 1}{1.048^{\frac{1}{12}} - 1}$, so the amount the bank needs to receive is $388,278.78.

   If the buyer pays this amount, we can find his annual effective yield by trying a few different yields:

Annual Effective Yield	Price Paid
5.400%	$387,854.69
5.300%	$392,075.87
5.385%	$388,483.88
5.395%	$388,064.24

   So the buyer's annual effective rate is 5.39%.

7. *Mrs. Young borrows $8,000 for one year at 7% simple interest. After 3 months, she repays $3,000.*

   (a) If the loan is calculate using the U.S. rule, how much does she need to pay 6 months after the start of the loan, to pay off the debt.

Using the U.S. rule, after 3 months, the balance outstanding is $8000(1 + \frac{0.07}{4}) = 8140$, so after her payment, the balance is \$5,140. After another 3 months, this has grown by a factor of 1.0175, to \$5,229.95, so this is the amount she needs to pay off the loan.

(b) She makes this payment, but the terms of the loan actually say that the Merchant's rule is the applicable rule. What happens at the end of the year, when the loan is due for settlement?

At the end of the year, the accumulated value of the loan is $8000 \times 1.07 = 8560$; the accumulated value of her first payment is $3000 \times 1.0475 = 3157.50$; and the accumulated value of her payment after 6 months is $5229.95 \times 1.035 = 5413.00$; so her payments total \$8555.50, and she is owed a return of \$10.50.