MATH 2600/STAT 2600, Theory of Interest
FALL 2014
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Homework Sheet 3
Model Solutions

1. *A loan of \$25,000 at $j_4 = 8\%$ is amortised with equal quarterly payments for 4 years.*

   (a) Calculate the quarterly payments. [Note: In the original version of this question, the payment frequency was unclear. I therefore provide solutions for both quarterly and monthly payments.]

   The quarterly payments are given by $\frac{25000}{a_{\overline{16}|0.02}} = \frac{25000\times 0.02}{1-1.02^{16}} = \$1,841.26$ (rounded up).

   For Monthly payments: The monthly payments are given by $\frac{25000}{a_{\overline{48}|1.02^{\frac{1}{3}}-1}} = \frac{25000\times(1.02^{\frac{1}{3}}-1)}{1-1.02^{16}} = \$609.71$ (rounded up).

   (b) Draw up a complete amortisation schedule for the loan.

Quarter	Outstanding Balance	Payment	Interest	Principal Repaid
1	25000	1841.26	500.00	1341.26
2	23658.74	1841.26	473.17	1368.09
3	22290.65	1841.26	445.81	1395.45
4	20895.21	1841.26	417.90	1423.36
5	19471.85	1841.26	389.44	1451.82
6	18020.03	1841.26	360.40	1480.86
7	16539.17	1841.26	330.78	1510.48
8	15028.69	1841.26	300.57	1540.69
9	13488.01	1841.26	269.76	1571.50
10	11916.51	1841.26	238.33	1602.93
11	10313.58	1841.26	206.27	1634.99
12	8678.59	1841.26	173.57	1667.69
13	7010.90	1841.26	140.22	1701.04
14	5309.86	1841.26	106.20	1735.06
15	3574.80	1841.26	71.50	1769.76
16	1805.03	1841.13	36.10	1805.03

   For Monthly payments:

Month	Outstanding Balance	Payment	Interest	Principal Repaid
1	$25,000.00	$609.71	$165.57	$444.14
2	$24,555.86	$609.71	$162.63	$447.08
3	$24,108.78	$609.71	$159.67	$450.04
4	$23,658.74	$609.71	$156.68	$453.03
5	$23,205.71	$609.71	$153.68	$456.03
6	$22,749.68	$609.71	$150.66	$459.05
7	$22,290.63	$609.71	$147.62	$462.09
8	$21,828.54	$609.71	$144.56	$465.15
9	$21,363.39	$609.71	$141.48	$468.23
10	$20,895.16	$609.71	$138.38	$471.33
11	$20,423.83	$609.71	$135.26	$474.45
12	$19,949.38	$609.71	$132.12	$477.59
13	$19,471.79	$609.71	$128.96	$480.75
14	$18,991.04	$609.71	$125.77	$483.94
15	$18,507.10	$609.71	$122.57	$487.14
16	$18,019.96	$609.71	$119.34	$490.37
17	$17,529.59	$609.71	$116.09	$493.62
18	$17,035.97	$609.71	$112.82	$496.89
19	$16,539.08	$609.71	$109.53	$500.18
20	$16,038.90	$609.71	$106.22	$503.49
21	$15,535.41	$609.71	$102.89	$506.82
22	$15,028.59	$609.71	$99.53	$510.18
23	$14,518.41	$609.71	$96.15	$513.56
24	$14,004.85	$609.71	$92.75	$516.96
25	$13,487.89	$609.71	$89.33	$520.38
26	$12,967.51	$609.71	$85.88	$523.83
27	$12,443.68	$609.71	$82.41	$527.30
28	$11,916.38	$609.71	$78.92	$530.79
29	$11,385.59	$609.71	$75.40	$534.31
30	$10,851.28	$609.71	$71.86	$537.85
31	$10,313.43	$609.71	$68.30	$541.41
32	$9,772.02	$609.71	$64.72	$544.99
33	$9,227.03	$609.71	$61.11	$548.60
34	$8,678.43	$609.71	$57.47	$552.24
35	$8,126.19	$609.71	$53.82	$555.89
36	$7,570.30	$609.71	$50.14	$559.57
37	$7,010.73	$609.71	$46.43	$563.28
38	$6,447.45	$609.71	$42.70	$567.01
39	$5,880.44	$609.71	$38.94	$570.77
40	$5,309.67	$609.71	$35.16	$574.55
41	$4,735.12	$609.71	$31.36	$578.35
42	$4,156.77	$609.71	$27.53	$582.18
43	$3,574.59	$609.71	$23.67	$586.04
44	$2,988.55	$609.71	$19.79	$589.92
45	$2,398.63	$609.71	$15.89	$593.82
46	$1,804.81	$609.71	$11.95	$597.76
47	$1,207.05	$609.71	$7.99	$601.72
48	$605.33	$609.34	$4.01	$605.33

2. *Mrs. Quine takes out a 25-year mortgage for a loan of \$1,400,000 at* $j_2 = 5\%$.

   (a) Calculate the monthly payments required.

   The monthly interest rate is $1.025^{\frac{1}{6}} - 1 = 0.004123915$, so payments are given by $\frac{1400000}{a_{\overline{300}|0.004123915}} = \frac{1400000 \times 0.004123915}{1-1.004123915^{-300}} = \$8,142.47$.

   (b) After 5 years, the interest rate increases to $j_2 = 5.5\%$, *calculate the new monthly payments if she wishes to keep the mortgage over 25 years.*

   The accumulated value of the payments she has made is $8142.47 s_{\overline{60}|0.004123915} = \$553,013.26$, and the value of the original loan is $1400000(1.004123915)^{60} = \$1,792,118.31$, so the outstanding balance is $1792118.31 - 553013.26 = \$1,239,105.04$.

   The new monthly interest rate is $1.0275^{\frac{1}{6}} - 1 = 0.004531682$, so the new payments are then $\frac{1239105.04}{a_{\overline{240}|0.004531682}} = \$8,480.33$.

   (c) If instead, she wishes to keep the mortgage payments the same, when will she finish paying off the mortgage?

   If she keeps the payments the same, we need to solve $8142.47 a_{\overline{n}|0.004531682} = 1239105.04$, or $\frac{1-1.004531682^{-n}}{0.004531682} = \frac{1239105.04}{8142.47}$, so $1-1.004531682^{-n} = 0.6896224371$, or $= 1.004531682^{-n} = 0.3103775629$, and $n = -\frac{\log(0.3103775629)}{\log(1.004531682)} = 258.8$, so she will need another 259 payments to finish paying off the mortgage. That is 19 months after the originally planned finishing date.

3. *Mrs. Roberts borrows \$8,000 for one year at 7% simple interest. After 3 months, she repays \$3,000. How much does she need to pay 6 months after the start of the loan, to pay off the debt.*

   (a) If the loan is calculated using the U.S. rule?

   If the loan is calculated using the U.S. rule, then after 3 months, she owes $8000\left(1 + \frac{0.07}{4}\right) = \$8,140$. She repays \$3,000, so the balance owing is \$5,140. After another 3 months, the balance is $5140\left(1 + \frac{0.07}{4}\right) = \$5,229.95$.

   (b) If the loan is calculated using the Merchant's rule?

   Using the Merchant's rule, at the end of the 6 month period, the original loan has a value of $8000\left(1 + \frac{0.07}{2}\right) = \$8,280$, while the payment of \$3,000 has a value of $3000\left(1 + \frac{0.07}{4}\right) = \$3,052.50$, so the balance owing is $8280 - 3052.50 = \$5,227.50$.

4. *Mr. Smith buys a cottage with a downpayment of \$50,000 and a 15-year mortgage for the remaining \$200,000 at* $j_2 = 4\%$. *There is a penalty of three times monthly interest on the outstanding balance for paying off the loan early. After 2 years, another company offers him a chance to refinance at* $j_2 = 3.4\%$ *for the remaining 13 years of the loan. Should he refinance?*

The original interest rate is $1.02^{\frac{1}{6}}-1 = 0.00330589$. The original mortgage payments are $\frac{150000}{a_{\overline{180}|0.00330589}} = \frac{150000\times 0.00330589}{1-1.00330589^{-180}} = \$1,107.06$. After two years, the accumulated value of the payments made is $1107.06 s_{\overline{24}|0.00330589} = \$27,604.47$, while the accumulated value of the debt is $150000(1.02)^4 = \$162,364.82$, so the outstanding balance is $162364.82-27604.47 = \$134,760.36$. The monthly interest is therefore $134760.36 \times 0.00330589 = \$445.50$, so three months' interest is 3 times this or \$1,336.51. If he refinances, the balance will therefore be \$136,096.86, and the new monthly interest rate will be $1.017^{\frac{1}{6}} - 1 = 0.00281347$, so the monthly payments will be $\frac{136096.86}{a_{\overline{156}|0.00281347}} = \frac{136096.86\times 0.00281347}{1-1.00281347^{-156}} = \$1,079.04$. Since the monthly payments have decreased, he should refinance.

5. *Mr. and Mrs. Thorpe buys a house in the US. They need to borrow \$300,000 at $j_{12} = 5.2\%$, amortised over 25 years. There is also a financing fee of \$5,000. What is the APR for this loan?*

   Since there is a fee of \$5,000, the actual loan amount is \$305,000. Amortising this over 25 years at $j_{12} = 5.2\%$ gives payments of $\frac{305000}{a_{\overline{300}|0.0043333333}} = \frac{305000\times 0.0043333333}{1-1.004333333^{-300}} = \$1,818.72$. The value of the loan they received is \$300,000, so we are looking for the monthly interest rate i such that $a_{\overline{300}|i} = \frac{300000}{1818.72} = 164.95$.

   We try a range of rates:

i	$a_{\overline{300}\|i}$
0.45%	164.44
0.445%	165.41
0.448%	164.82
0.4475%	164.92
0.4473%	164.96

   We therefore see that the APR has monthly interest between 0.4473% and 0.4475%, so $j_{12} = 5.37\%$. This is an annual effective rate of 5.50%.

6. *A bank lends \$500,000 to Mrs. Underhill. The loan is payed back with monthly interest-only payments at $j_{12} = 6\%$, with the principal returned as a lump sum after 20 years. After 8 years, the bank sells the loan to a private investor, who wishes to achieve an annual effective yield of 5.4%.*

   (a) How much does the investor pay for the loan?

   We use Makeham's formula: The present value of the final payment at an annual effective rate of 5.4%is $K = 500000(1.054^{-12} = \$266,000.64$. The monthly rate that gives an annual effective rate of 5.4% is $(1.054)^{\frac{1}{12}} - 1 = 0.439\%$ so $P = K + (F - K)\frac{0.5}{0.4392322} = 266000.46 + 233999.54\frac{0.5}{0.439} = \$532,373.82$.

   (b) If the bank wants to make an annual effective return of 5.3% on its investment, what annual effective yield would the buyer receive?

   (i) 6.21%

(ii) 6.89%

(iii) 7.24%

(iv) 7.53%

In order to make an annual effective return of 5.3%, the value of the bank's initial investment after 8 years, should be $500000(1.053)^8 = \$755,782.75$, and the value of the monthly payments of \$2,500 that the bank has received is $2500s_{\overline{96}|0.004312877} = 2500\frac{1.004312877^{96}-1}{0.004312877} = \$296,533.82$, so to achieve the desired yield, the bank must receive $755782.75 - 296533.82 = \$459,248.93$. If the buyer pays this much for the loan, we can calculate his annual effective yield by trying the values above

Annual effective rate	monthly interest rate	price for loan
6.21%	0.5033298%	\$498,297.51
6.89%	0.5567951%	\$471,924.85
7.24%	0.5841925%	\$459,087.38
7.53%	0.6068312%	\$448,809.81

So we see that 7.24% is the effective yield.

7. *Mrs. Vickers borrows \$300,000 to invest in the stock market. She has two options for repayment. She can either amortise the loan over 25 years at $j_{12} = 4\%$, or she can make interest only payments at $j_{12} = 4.4\%$ for 25 years, then pay off the balance with a lump sum payment at the end of the 25 years. What is the smallest rate of return she needs on her investments to make the interest-only payments the better deal?*

   (i) 4.80%

   (ii) 5.40%

   (iii) 5.76%

   (iv) 6.04%

   If she amortises the loan over 25 years, her monthly payments are $\frac{300000}{a_{\overline{300}|0.00333333}} = \frac{300000\times 0.0033333333}{1-1.00333333^{-300}} = \$1,583.51$. If her monthly rate of return is i, then by the end of 25 years, the amount she has is $300000(1+i)^{300} - 1583.51s_{\overline{300}|i}$. On the other hand, if she takes the interest-only loan, her monthly payments are \$1,100, so the amount she has after 25 years when she pays off the loan is $300000(1+i)^{300} - 1100s_{\overline{300}|i} - 300000$. We want to find the value of i that makes these two outcomes equal. That is $300000(1+i)^{300} - 1583.51s_{\overline{300}|i} = 300000(1+i)^{300} - 1100s_{\overline{300}|i} - 300000$, which gives $483.51s_{\overline{300}|i} = 300000$, so $s_{\overline{300}|i} = \frac{300000}{483.51} = 620.46$.

   We try the values given:

Annual effective rate	monthly interest rate i	$s_{\overline{300}\|i}$
4.80%	0.391%	569.34
5.40%	0.439%	620.18
5.76%	0.468%	653.18
6.04%	0.490%	680.23

So the rate of return needed is 5.40%.