MATH 3030, Abstract Algebra
FALL 2012
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Sample Midterm

This sample midterm is deliberately longer than the actually midterm, to better cover the range of possible questions that could be asked. I have provided estimated times for each question. Based on similar estimated times, it should be possible to complete the midterm examination in 40 minutes (out of the 50 available).

Basic Questions

1. Are the following multiplication tables groups? Justify your answers. [5 mins]

(a)

	a	b	c
a	c	a	c
b	a	b	a
c	c	a	c

(b)

	a	b	c	d	e
a	a	b	c	d	e
b	b	a	e	c	d
c	c	d	a	e	b
d	d	e	b	a	c
e	e	c	d	b	a

(c)

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

2. Which of the following are groups: [6 mins]

(a) $\mathbb{N} = \{n \in \mathbb{Z} | n \geqslant 0\}$ with the operation $a * b$ given by addition without carrying, that is, write a and b (in decimal, including any leading zeros necessary) and in each position add the numbers modulo 10, so for example $2456 * 824 = 2270$.

(b) The set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 0$ with pointwise addition (i. e. $(f + g)(x) = f(x) + g(x)$).

(c) The set of real numbers with the operation $x * y = \frac{xy}{x+y}$.

3. How many generators are there in the cyclic group $\mathbb{Z}_{28}$? [2 mins]

4. Which of the following are subgroups of $\mathbb{Z} \times \mathbb{Z}$? [15 mins]

(a) The set of all pairs (a, b) where a is divisible by 6.

(b) The set of all pairs (a, b) such that $a + 3b = 0$.

(c) The set of all pairs (a, b) such that $2a + b = 2$.

(d) The set of all pairs (a, b) such that $5a + 2b$ is divisible by 4.

(e) The set of all pairs (a, b) such that $a^2 + b^2$ is a square number (i.e. $a^2 + b^2 = c^2$ for some $c \in \mathbb{Z}$.)

(f) The set of all pairs (a, b) such that $a \geqslant b$.

5. Which of the following are subgroups of the group of permutations of the 6 element set $\{1, 2, 3, 4, 5, 6\}$? [5 mins]

(a) The set of permutations σ such that $\sigma(1) + \sigma(4) + \sigma(5) = 10$.

(b) The set of permutations σ that either fix the set of odd numbers of send it to the set of even numbers. That is: either $\sigma(\{1, 3, 5\}) = \{1, 3, 5\}$ or $\sigma(\{1, 3, 5\}) = \{2, 4, 6\}$.

6. (a) Describe the subgroup of $\mathbb{Z} \times \mathbb{Z}_{12}$ generated by $(2, 8)$. [2 mins]

(b) Describe the subgroup of $\mathbb{Z} \times \mathbb{Z}_{12}$ generated by $(2, 8)$ and $(3, 4)$. [2 mins]

7. (a) Write $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 9 & 7 & 6 & 8 & 1 & 3 \end{pmatrix}$ as a product of disjoint cycles. [2 mins]

(b) What is the order of σ? [1 min]

(c) Is σ an odd or even permutation? [2 mins]

(d) Which of the following permutations are conjugate to σ in S_9? [3 mins]

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 9 & 3 & 8 & 1 & 7 & 6 & 4 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 7 & 9 & 1 & 8 & 4 & 6 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 1 & 5 & 6 & 8 & 9 & 7 \end{pmatrix}$

8. Draw the Cayley graph of A_4 with generators (123) and (234). [8 mins]

9. Which of the following subgroups are normal? [14 mins]

(a) The subgroup of the group of symmetries of a hexagon generated by a 120° rotation.

(b) The subgroup of the group of symmetries of a hexagon generated by a 180° rotation.

(c) The subgroup of the additive group of real numbers generated by the numbers whose square is rational.

(d) The subgroup of the multiplicative group of all invertible 3×3 matrices with real coefficients consisting of matrices with rational determinant.

10. Find the index of $\langle (1,4),(5,7) \rangle$ in $\mathbb{Z} \times \mathbb{Z}$. [5 mins]

11. Is there a transitive permutation group on 4 elements in which every element has order less than 4? [2 mins]

12. Which of the following functions are homomorphisms. [5 mins]

    (a) $f : S_6 \to S_3$ given by $f(\sigma)(1) = \sigma(1) + \sigma(4)(\pmod 3)$ $f(\sigma)(2) = \sigma(2) + \sigma(5)(\pmod 3)$ $f(\sigma)(3) = \sigma(3) + \sigma(6)(\pmod 3)$

    (b) $f : D_6 \to D_3$ given by $f(x) = x$ if x preserves the triangles formed by alternating vertices of the hexagon, and $f(x)$ is x followed by a $180°$ rotation otherwise.

13. (a) Calculate the commutator subgroup of $\mathbb{Z} \times S_3$. [3 mins]

    (b) Calculate the factor group of $\mathbb{Z} \times S_3$ over its commutator subgroup. [3 mins]

14. Calculate the centre of $S_3 \times \mathbb{Z}_6$. [5 mins]

Theoretical Questions

15. Prove that the intersection of two subgroups of a group is another subgroup. [5 mins]

16. Show that any finite group of even order has an element of order 2. [Hint: Suppose all non-identity elements have order at least 3. Now partition the group into a collection of disjoint pairs and the identity element.] [5 mins]

17. Let G be a permutation group on a finite set with orbits of sizes $a_1, \ldots, a_m$. Show that $|G|$ is at least the lowest common multiple of $a_1, \ldots, a_m$. [5 mins]

18. State and prove Lagrange's theorem about the order of a subgroup of a finite group. [5 mins]

19. Show that for subgroups $H \leqslant K \leqslant G$, if $(G : K)$ and $(K : H)$ are finite, then $(G : H) = (G : K)(K : H)$. [5 mins]

20. Let H be a subgroup of G. Show that $N_G(H) = \{x \in G | xHx^{-1} = H\}$ is the largest subgroup of G which contains H as a normal subgroup. [5 mins]

21. Show that the composite of two group homomorphisms is another group homomorphism. [4 mins]

22. Let $H \leqslant G$. Show that the commutator subgroup of H is a subgroup of the commutator subgroup of G, and that the centre $Z(H)$ contains $Z(G) \cap H$. [6 mins]