MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 15
Due: Friday 22nd March: 3:30 PM

Basic Questions

1. Find a basis for the splitting field over $\mathbb{Q}$ of $x^3 - 4$.
2. (a) What is the order of $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$?

   (b) What is the order of $G(\mathbb{Q}(\sqrt[3]{2}, \frac{sqrt3}{2}i)/\mathbb{Q}(\sqrt{3}2i))$?
3. Find an element α such that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3}) = \mathbb{Q}(\alpha)$, and express $\sqrt{2}$ and $\sqrt[3]{3}$ as polynomials in this α over $\mathbb{Q}$.

Theoretical Questions

4. Show that if E is a finite extension of F, and E is a splitting field over F, then E is the splitting field of a single polynomial over F.
5. Show that if E is a splitting field over F, then for any element $\alpha \in E$, E contains all conjugates of α over F.
6. Let E be a splitting field of an irreducible polynomial $f(x)$ over F. Let σ be an automorphism of E that leaves F fixed.

   (a) Show that σ induces a permutation of the zeros of $f(x)$.

   (b) Show that if σ' is another automorphism of E that leaves f fixed and induces the same permutation on the zeros of $f(x)$ as σ, then $\sigma' = \sigma$.
7. Show that if E is an algebraic extension of a perfect field F, then E is perfect.
8. Let K be a field extension of F, and let L be a field extension of K. Let $\alpha \in L$ be algebraic over F. Show that $[K(\alpha) : K] \leqslant [F(\alpha) : F]$.

Bonus Questions

9. For an infinite algebraic field extension, we will say that the extension is separable if every element of the larger field is separable over the smaller field. Show that if E is a separable extension of F and K is a separable extension of E, then K is a separable extension of F.