MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 9
Due: Friday 30th November: 3:30 PM

Basic Questions

1. Factorise $x^4 + 3x^3 + 2x^2 + 9x - 3$:

   (a) over $\mathbb{Z}_3$.

   (b) over $\mathbb{Z}_6$.

   (c) over $\mathbb{Z}$.

2. Show that $f(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Z}$. [Hint: consider $x = y + 1$ and use Eisenstein's criterion.]

3. Find all solutions to the equation $x^2 + 2x - 3 = 0$ in $\mathbb{Z}_{21}$.

4. Find all prime numbers p such that $x - 4$ is a factor of $x^4 - 2x^3 + 3x^2 + x - 2$ in $\mathbb{Z}_p[x]$.

5. Find a generator for the multiplicative group of non-zero elements of $\mathbb{Z}_{19}$.

6. Show that $f(x) = x^2 + 3x + 2$ does not factorise uniquely over $\mathbb{Z}_6$.

7. Show that $f(x) = x^3 + 4x^2 + 1$ is irreducible in $\mathbb{Z}_7$. [Hint: if it is not irreducible then it must have a linear factor.]

Theoretical Questions

8. Show that if D is an integral domain, then so is $D[x]$.

9. Let R be a ring. (a) Show that the ring of functions from R to R is a ring with pointwise addition and multiplication. That is:

$$(f+g)(x) = f(x) + g(x)$$
$$fg(x) = f(x)g(x)$$

   (b) Show that the set of all functions describable by polynomials gives a subring of the ring of all functions.

   (c) Show that this ring is not always isomorphic to the polynomial ring $R[x]$. [Hint: let R be a finite field $\mathbb{Z}_p$ for some prime p.]

10. Show that the remainder when a polynomial $f(x) \in F[x]$ is divided by $x - a$ is $f(a)$.