MATH/STAT 3360, Probability
FALL 2012
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Midterm Examination
Model Solutions

1. *What is the probability that a five-card poker hand is a full house (three cards of one rank and two of another)?*

   There are $\binom{52}{5}$ possible poker hands. There are $13 \times 12 \times \binom{4}{3} \times \binom{4}{2}$ full houses, so the probability of a full house is

$$\frac{13 \times 12 \times 4 \times 6}{\binom{52}{5}} = \frac{13 \times 12 \times 6 \times 4 \times 5!}{52 \times 51 \times 50 \times 49 \times 48} = \frac{2 \times 3}{17 \times 5 \times 49} = 0.00144$$

2. *A fair coin is tossed 11 times. What is the probability that the sequence THTHTHT occurs somewhere in the 11 tosses?*

   The sequence THTHTHT can occur starting in positions 1, 2, 3, 4 or 5. Let the events A_1, A_2, A_3, A_4, A_5 be the event that it occurs starting in positions 1, 2, 3, 4, 5 respectively. We have that $P(A_i) = \frac{1}{2^7}$, while $P(A_1 \cap A_3) = P(A_2 \cap A_4) = P(A_3 \cap A_5) = \frac{1}{2^9}$ and $P(A_1 \cap A_5) = P(A_1 \cap A_3 \cap A_5) = \frac{1}{2^{11}}$, with all other intersections empty. We therefore get that $P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = \frac{5}{2^7} - \frac{3}{2^9} - \frac{1}{2^{11}} + \frac{1}{2^{11}} = \frac{17}{2^9} = 0.0332$.

3. *Three fair 6-sided dice are rolled. Are the following events independent?*

   (i) At least two of the dice show the same number

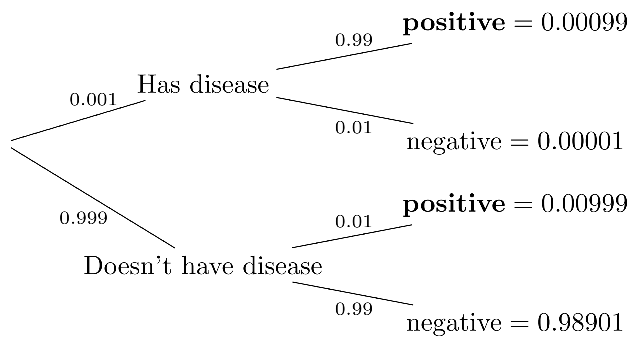
   (ii) The total is 10.

   Let A be the event "At least two of the dice show the same number" and B be the event "The total is 10". We have that $P(A) = \frac{3\times6\times5+6}{6^3} = \frac{4}{9}$, $P(B) = \frac{4+5+6+5+4+3}{6^3} = \frac{1}{8}$, and $P(A \cap B) = \frac{9}{6^3} = \frac{1}{24}$. Now $P(A)P(B) = \frac{1}{18} \neq P(A \cap B)$, so A and B are not independent.

4. *A patient is given a routine test for a rare disease. The disease affects 1 person in 1000. The test is 99% accurate, so there is only a 1% chance of giving the wrong result. The test result is positive (i.e. indicates the patient has the disease).*

   What is the probability that the patient actually has the disease?

   In a tree diagram:



So the total probability of a positive test result is $0.00099 + 0.00999 = 0.01098$, so the probability that the patient has the disease given a positive test result is $\frac{0.00099}{0.01098} = 0.09016$.

5. *A company is conducting a survey. They want to determine the proportion of people who would buy their new product. If the true proportion is 30%, how many people do they need to survey so that the probability that their estimate is within 2% of the true value (i.e. between 28% and 32%) is at least 95%? [You may use any reasonable approximation for the distribution of the number of people who say they would buy this product. You may also assume that the total number of people who could be surveyed is large enough that different peoples responses are independent.]*

   If they survey n people, the number, N, who say they will buy the product is a binomial random variable with parameters n and 0.3. We can approximate this as a normal random variable with mean $0.3n$ and variance $0.21n$. We want to ensure that $P(0.28n < N < 0.32n) \geqslant 0.95$. Using the normal approximation, we consider $Z = \frac{N-0.3n}{\sqrt{0.21n}}$, which is normally distributed with mean 0 and variance 1. We have

$$P(0.28n < N < 0.32n) = P\left(\frac{-0.02n}{\sqrt{0.21n}} < Z < \frac{0.02n}{\sqrt{0.21n}}\right) = \Phi\left(\frac{0.02n}{\sqrt{0.21n}}\right) - \Phi\left(\frac{-0.02n}{\sqrt{0.21n}}\right) = 2\Phi\left(\frac{0.02n}{\sqrt{0.21n}}\right)$$

   We see that $2\Phi(1.96) = 0.975$, so we need $\frac{0.02\sqrt{n}}{\sqrt{0.21}} = 1.96$ or $\sqrt{n} = 44.909$, which gives $n = 2016.84$, so they need to survey 2017 people.

6. *A motor on a boat consists of three blades arranged in a circle. The blades are equally spaced, and each covers an angle of $30°$. If the motor passes over a small rock, what is the probability that one of the blades hits the rock?*

   Let X be the angle from the start of one of the blades to the rock. The blades hit the rock if $0 < X < 30$ or $120 < X < 150$ or $240 < X < 270$. The total probability of one of these is $\frac{30+30+30}{360} = \frac{1}{4}$.

7. *A company makes light bulbs. The company has two machines for making them. light bulbs made by one machine have lifetime (in years) exponentially distributed with parameter 0.2, and light bulbs made by the other machine have lifetime exponentially distributed with parameter 0.3. 40% of its light bulbs are made by the first machine.*

   (a) What is the probability that a randomly chosen light bulb from this company lasts for at least 1 year?

   The probability that a bulb from the first machine lasts at least one year is $e^{-0.2}$, and the probability that a bulb from the second machine lasts at least a year is $e^{-0.3}$. The probability that a randomly chosen bulb lasts at least a year is therefore $0.4e^{-0.2} + 0.6e^{-0.3} = 0.772$.

   (b) Given that it lasts for 1 year, what is the probability that it was produced by machine 1?

   The probability that it was produced by machine 1 and lasts for at least 1 year is $0.4e^{-0.2}$, so the probability that it was produced by the first machine, given that it lasts for a year is $\frac{0.4e^{-0.2}}{0.4e^{-0.2}+0.6e^{-0.3}} = 0.4242$.

   (c) Given that a light bulb has lasted for 1 year, what is the probability that it lasts for a second year?

   By the memoryless property of the exponential distribution, the probability of lasting for another year is $e^{-0.2}$ if the bulb is from the first machine and $e^{-0.3}$ if it is from the second. That is, the probability that it lasts for another year is $\frac{0.4e^{-0.2}}{0.4e^{-0.2}+0.6e^{-0.3}}e^{-0.2} + \frac{0.6e^{-0.3}}{0.4e^{-0.2}+0.6e^{-0.3}}e^{-0.3} = 0.7739$.

8. *The number of customers a company has on a given day is a Poisson random variable with parameter 5.5.*

   (a) What is the probability that the company receives exactly 6 customers on a given day?

   Let N be the number of customers. $P(N = 6) = e^{-5.5}\frac{5.5^6}{6!} = 0.1571$.

   (b) For each customer, the company receives $200, but it can handle a maximum of 10 customers per day. What is the expected amount received in a given day?

n	$P(N=n)$	amount received	$200nP(N=n)$
0	$e^{-5.5} = 0.004086771$	0	0
1	$5.5e^{-5.5} = 0.022477243$	200	4.49544858231047
2	$\frac{5.5^2}{2}e^{-5.5} = 0.061812418$	400	24.7249672027076
3	$\frac{5.5^3}{3!}e^{-5.5} = 0.113322766$	600	67.9936598074459
4	$\frac{5.5^4}{4!}e^{-5.5} = 0.155818804$	800	124.655042980318
5	$\frac{5.5^5}{5!}e^{-5.5} = 0.171400684$	1000	171.400684097937
6	$\frac{5.5^6}{6!}e^{-5.5} = 0.157117294$	1200	188.54075250773
7	$\frac{5.5^7}{7!}e^{-5.5} = 0.123449302$	1400	172.829023132086
8	$\frac{5.5^8}{8!}e^{-5.5} = 0.084871395$	1600	135.794232460925
9	$\frac{5.5^9}{9!}e^{-5.5} = 0.051865853$	1800	93.3585348168858
10 or more	$e^{-5.5} = 0.05377746$	2000	107.554939032732

So the expected amount received is the sum of the last column, which is \$1091.35.