ACSC/STAT 3703, Actuarial Models I

WINTER 2023
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Practice Midterm Examination

Model Solutions

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *An individual has utility function $u(x) = \frac{x}{x+a}$ for some constant a. The individual has wealth \$42,000 and is willing to pay up to \$142 to insure against a loss of \$23,000 with probability 0.006. How much would she be willing to pay to insure against a loss of \$19,000 with probability 0.064? [15 mins]*

   Since she is willing to pay up to \$142 to insure against a loss of \$23,000 with probability 0.006, we know that her expected utility without insurance is equal to her utility after paying \$142. That is

$$u(41858) = 0.006u(19000) + 0.994u(42000)$$

   Substituting the form for $u(x)$, this becomes

$$\frac{41858}{41858 + a} = 0.006\frac{19000}{19000 + a} + 0.994\frac{42000}{42000 + a}$$

   We multiply through to solve for a:

$$\begin{aligned}
41858(19000 + a)(42000 + a) &= 114(41858 + a)(42000 + a) + 41748(41858 + a)(19000 + a)\\
41858a^2 + 2553338000a + 33402684000000 &= 41862a^2 + 2550259596a + 33402684000000\\
4a^2 &= 3078404a\\
a &= 769601
\end{aligned}$$

   Now for a loss of \$19,000 with probability 0.064, the expected utility without insurance is

$$0.064\frac{23000}{23000 + 769601} + 0.936\frac{42000}{42000 + 769601} = 0.0502947708969$$

   Therefore, she would be willing to pay a premium such that her resulting wealth had utility 0.0502947708969. That is

$$\begin{aligned}
\frac{w}{w + 769601} &= 0.0502947708969\\
w &= 0.0502947708969(w + 769601)\\
0.949705229103w &= 38706.905977\\
w &= 40756.7577716
\end{aligned}$$

   This is a premium of $42000 - 40756.7577716 = \$1243.24$.

2. *An individual has wealth \$132,000. The individual has utility function of the form $u(x) = a\log(x) - \frac{1}{x}$ for some value of a.*

   The individual elects not to purchase insurance to cover a loss of \$22,000 with probability 0.014 for a premium of \$322. What is the most the individual might be willing to pay for insurance covering a loss of \$18,000 with probability 0.01?

   (i) \$193.40

   (ii) \$208.09

   (iii) \$220.51

   (iv) \$232.00

   [15 mins]

   Since the individual did not insure the loss of \$22,000, we have

$$u(131678) \leqslant 0.014u(111000) + 0.986u(132000)$$

   Substituting the parametric form for $u(x)$, this becomes

$$\begin{aligned} a\log(131678) - \frac{1}{131678} &\leqslant 0.014a\log(111000) - \frac{0.014}{111000} + 0.986a\log(132000) - \frac{0.986}{132000} \\ 2.14007 \times 10^{-5}a &\leqslant 5.1829531351 \times 10^{-16} \\ a &\leqslant 2.42186149757 \times 10^{-11} \end{aligned}$$

   They would be willing to pay x to insure the loss of \$18,000 if

$$u(132000 - x) > 0.01u(114000) + 0.99u(132000)$$

   Substituting the form for $u(x)$ gives

$$\begin{aligned} a\log(132000 - x) - \frac{1}{132000 - x} &> 0.01a\log(114000) - \frac{0.01}{114000} + 0.99a\log(132000) - \frac{0.99}{132000} \\ &= 11.7890911669a - 7.58771929825 \times 10^{-6} \\ (\log(132000 - x) - 11.7890911669)a &> \frac{1}{132000 - x} - 7.58771929825 \times 10^{-6} \end{aligned}$$

   If $132000 - x > e^{11.7890911669} = 131806.625205$, or $x < 193.374795$ then the left-hand side is positive. The right-hand side is positive when $\frac{1}{132000-x} > 7.58771929825 \times 10^{-6}$ which happens when $x < 208.092486$. Thus the inequality only holds if $x < 193.374795$, in which case, the left-hand side is an increasing function of a, so the amount the individual would be willing to pay if maximised when a is as large as possible. Thus, we substitute $a = 2.42186149757 \times 10^{-11}$ to get

$$\begin{aligned} 2.42186149757 \times 10^{-11}(\log(132000 - x) - 11.7890911669) &> \frac{1}{132000 - x} - 7.58771929825 \times 10^{-6} 2.42186149757 \times 10^{-11}\log(1 \\ \log(132000 - x) &> \frac{41290552783.6}{132000 - x} - 313289.3351 \end{aligned}$$

We test this for the given values:

x	$132000-x$	$\log(132000-x)-\frac{41290552783.6}{132000-x}+313289.3351$
(i) \$193.40	131806.60	34.92
(ii) \$208.09	131791.91	0.005797
(iii) \$220.51	131779.49	-29.52
(iv) \$232.00	131768.00	-56.84

Thus, we see that (ii) $x = \$208.09$ is the largest premium that they might be willing to pay.

3. *Which of the following risks are insurable? For risks which are not insurable, explain why they are not insurable.*

   (i) The risk that a car will need to be replaced in 20 years time.

   (ii) The risk that an individual is abducted by aliens.

   (iii) The risk that a planned holiday will need to be cancelled.

   (iv) The risk that an individual will feel ill.

   (v) The risk that a banknote will be torn.

   (vi) The risk to a professional sports team that its star player will be injured and unable to play.

   (vii) The risk that an individual is not selected for a particular job.

   (viii) The risk that all civilisations on earth will be destroyed.

   [10 mins]

   (i) This is not insurable as the loss is not random.

   (ii) This is not insurable as there is not sufficient data to estimate the probability.

   (iii) This is an insurable risk. Indeed, this kind of agricultural insurance is sold.

   (iv) This is not insurable as the loss is not well-defined.

   (v) This is not insurable as it is not economically feasible.

   (vi) This is an insurable risk.

   (vii) This is not insurable as the risks are not homogeneous.

   (viii) This is not insurable as losses from different policyholders are not independant.

4. *A homeowner's house is valued at \$560,000. However, the home is insured only to a value of \$360,000. The insurer requires 80% coverage for full insurance. The home sustains \$6,000 of fire damage. The deductible is \$5,000, decreasing linearly to zero for losses of \$8,000. How much does the insurer reimburse? [5 mins]*

   The insurer pays $\frac{360000}{560000\times 0.8} = 80.3571428571\%$ of the costs. For a loss of \$6,000, the deductible is $5000 \times \frac{2}{3}$, so the insurer pays $\left(6000 - \frac{10000}{3}\right) \times 0.803571428571 = \$2,142.86$.

5. *A marine insurance policy includes a deductible of \$10,000, a policy limit of \$5,000,000 and co-insurance of 20% payable by the policyholder. If the co-insurance is applied before the policy limit, how much would the insurer reimburse for a loss of*

   (i) \$5,000

(ii) \$15,000

(iii) \$5,200,000

(iv) \$10,000,000
[5 mins]

(i) \$5,000 is less than the deductible, so nothing would be reimbursed.

(ii) The insurance would reimburse $(15000 - 10000) \times 0.8 = \$4,000$.

(iii) The insurance would reimburse $(5200000 - 10000) \times 0.8 = \$4,152,000$.

(iv) The insurance would reimburse \$5,000,000, as the amount reimbursed would exceed the policy limit.

6. *An auto insurance company uses an expected loss ratio of 0.81. In accident year 2014, the earned premiums were \$1,420,000. In 2014, the insurance company made a total of \$189,300 in loss payments for accident year 2014, a total of \$152,500 in 2015, and a total of \$239,600 in 2016. What loss reserves should the company hold for this accident year at the end of 2016.*

   [5 mins]

   The total loss payments made are \$581,400. The expected total payments are $1420000 \times 0.81 = \$1,150,200$. Therefore the reserve should be $1150200 - 581400 = \$568,800$.

7. *The following table shows the cumulative losses (in thousands) on claims from one line of business of an insurance company over the past 6 years.*

	Development year					
Accident year	*0*	*1*	*2*	*3*	*4*	*5*
2011	*751*	*1,022*	*1,448*	*1,133*	*1,473*	*1,493*
2012	*1,337*	*1,297*	*1,460*	*1,537*	*1,679*	
2013	*1,250*	*1,624*	*1,815*	*1,860*		
2014	*1,325*	*1,512*	*1,685*			
2015	*1,471*	*1,536*				
2016	*2,036*					

Using the average for calculating loss development factors, esimate the total reserve needed for payments to be made in 2018 using.

(a) The loss development triangle method [15 mins]

We calculate the following loss development factors:

Development year	Loss Development Factor
1/0	$\frac{1}{5}\left(\frac{1022}{751}+\frac{1297}{1337}+\frac{1624}{1250}+\frac{1512}{1325}+\frac{1536}{1471}\right) = 1.16309083475$
2/1	$\frac{1}{4}\left(\frac{1448}{1022}+\frac{1460}{1297}+\frac{1815}{1624}+\frac{1685}{1512}\right) = 1.19363330156$
3/2	$\frac{1}{3}\left(\frac{1133}{1448}+\frac{1537}{1460}+\frac{1860}{1815}\right) = 0.95333055933$
4/3	$\frac{1}{2}\left(\frac{1473}{1133}+\frac{1679}{1537}\right) = 1.19623801482$
5/4	$\frac{1493}{1473} = 1.01357773252$

Using the loss development triangle method, the cumulative payments up to 2017 are

$$1679 \times 1.01357773252 + 1860 \times 1.19623801482 + 1685 \times 0.95333055933 + 1536 \times 1.19363330156 + 2036 \times 1.16309083475 = 9734.63540369$$

The cumulative payments up to 2018 are

$$\begin{aligned}1679 \times 1.01357773252 + 1860 \times 1.19623801482 \times 1.01357773252 + 1685 \times 0.95333055933 \times 1.19623801482 + \\ 1536 \times 1.19363330156 \times 0.95333055933 + 2036 \times 1.16309083475 \times 1.19363330156 = 10453.0443718\end{aligned}$$

The payments to be made in 2018 are therefore $10453.0443718 - 9734.63540369 = \$718,408.97$.

(b) The Bornhuetter-Fergusson method. The expected loss ratio is 0.76 and the earned premiums in each year are given in the following table:

Year	*Earned Premiums (000's)*
2011	*1943*
2012	*2430*
2013	*2623*
2014	*2804*
2015	*3356*
2016	*3673*

[15 mins]

Under the Bornhuetter-Fergusson method, the proportion of total payments in each year is given by:

Dev. Year	Proportion of total payments
0	$\frac{1}{1.013577733\times1.196238015\times0.953330559\times1.193633302\times1.163090835} = 0.623156693912$
1	$\frac{1}{1.013577733\times1.196238015\times0.953330559\times1.193633302} - \frac{1}{1.013577733\times1.196238015\times0.953330559\times1.193633302\times1.163090835} = 0.101631145394$
2	$\frac{1}{1.013577733\times1.196238015\times0.953330559} - \frac{1}{1.013577733\times1.196238015\times0.953330559\times1.193633302} = 0.140343062249$
3	$\frac{1}{1.013577733\times1.196238015} - \frac{1}{1.013577733\times1.196238015\times0.953330559} = -0.040375175282$
4	$\frac{1}{1.013577733} - \frac{1}{1.013577733\times1.196238015} = 0.161848426438$
5	$1 - \frac{1}{1.013577733} = 0.013395847289$

This gives us:

Year	Expected total losses	Expected losses in 2018
2013	1993.48	$1993.48 \times 0.013395847289 = 26.7043536537$
2014	2131.04	$2131.04 \times 0.161848426438 = 344.905470676$
2015	2550.56	$2550.56 \times -0.040375175282 = -102.979307067$
2016	2791.48	$2791.48 \times 0.140343062249 = 391.764851407$

The total reserves needed for payments in 2018 are therefore $26.7043536537 + 344.905470676 - 102.979307067 + 391.764851407 = \$660,395.37$.

8. *An insurance company starts a new line of insurance in 2016, and collects a total of \$1,900,000 in premiums that year, and the estimated incurred losses for accident year 2016 are \$1,384,000. Half of the premium payments are made at the beginning of the year, and the other half are uniformly distributed over the year. An actuary is using*

this data to estimate rates for premium year 2018. Claims are subject to 4% inflation per year. By what percentage should premiums increase from 2016 in order to achieve a loss ratio of 0.75? [15 mins]

The earned premiums for 2016 are $1900000\left(\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\right) = \$1,475,000$. [Half the policies earn the full premium in 2016, while the other half earn on average half of the premium collected in 2016.] This means that the loss ratio is $\frac{1384000}{1425000} = 0.97122807018$. For losses in accident year 2016, the average inflation from the start of 2016 to the time of loss is given by $\int_0^1 \frac{2}{3}(t+1)(1.04)^t\,dt$. [The proportion of policies in force at time t during 2016 is proporional to $t+1$. The $\frac{2}{3}$ is a normalisation constant to get the pdf of the time of a random loss.] We calculate

$$\begin{aligned}
\int_0^1 \frac{2}{3}(t+1)(1.04)^t\,dt &= \frac{2}{3}\int_0^1 (t+1)e^{\log(1.04)t}\,dt\\
&= \frac{2}{3}\left(\int_0^1 te^{\log(1.04)t}\,dt + \left[\frac{e^{\log(1.04)t}}{\log(1.04)}\right]_0^1\right)\\
&= \frac{2}{3}\left(\left[\frac{te^{\log(1.04)t}}{\log(1.04)}\right]_0^1 - \int_0^1 \frac{e^{\log(1.04)t}}{\log(1.04)}\,dt + \frac{0.04}{\log(1.04)}\right)\\
&= \frac{2}{3}\left(\frac{1.04}{\log(1.04)} - \left[\frac{e^{\log(1.04)t}}{\log(1.04)^2}\right]_0^1 + \frac{0.04}{\log(1.04)}\right)\\
&= \frac{2}{3}\left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2} + \frac{0.04}{\log(1.04)}\right)\\
&= 1.02209143289
\end{aligned}$$

For policy year 2018, the average inflation from the start of the year to the time of loss is

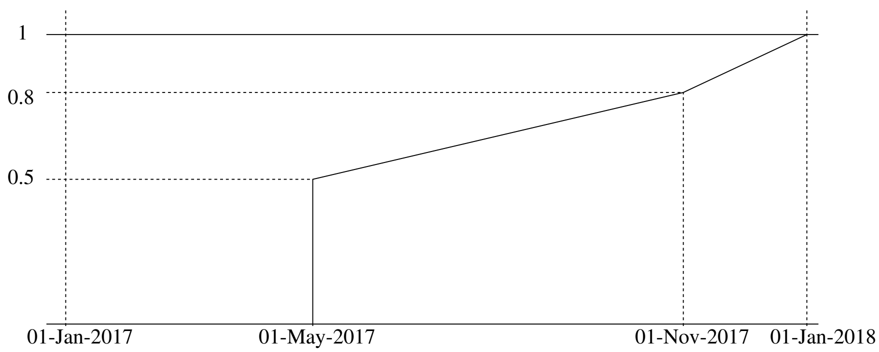
$$\begin{aligned}
\int_0^1 te^{\log(1.04)t}\,dt + 1.04\int_0^1 (1-t)e^{\log(1.04)t}\,dt &= 1.04\int_0^1 e^{\log(1.04)t}\,dt - 0.04\int_0^1 te^{\log(1.04)t}\,dt\\
&= 1.04\times\frac{0.04}{\log(1.04)} - 0.04\left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right)\\
&= 1.04013332308
\end{aligned}$$

The inflation from accident year 2016 to policy year 2018 is therefore $(1.04)^2 \times \frac{1.04013332308}{1.02209143289} = 1.10069233147$. The increase in premium is therefore $\frac{0.97122807018}{0.75}\times 1.10069233147 = 1.42536438527$, so a 42.54% increase in premium is required.

9. *In 2021, a home insurer collected $22,480,000 in earned premiums, and paid $19,380,000 in payments. There was a rate change on 1st May 2021. Before the rate change, the premium was $790. After the rate change, the premium was $830. Ignoring inflation, what should the new premium be to achieve an expense ratio of 20%? [15 mins]*

   The new premium was sold for the final $\frac{8}{12}$ of 2021. Therefore, it applies to $\frac{1}{2}\times\left(\frac{8}{12}\right)^2 = \frac{2}{9}$ of the earned premiums in 2021. Therefore, if we adjust the earned premiums for 2021 to the new premium, we get $22480000 \times \frac{830}{\frac{7}{9}\times 790 + \frac{2}{9}\times 830} = 23355438.1085$. For these adjusted premiums, the loss ratio is $\frac{19380000}{23355438.1085} = 0.829785333504$. Therefore, the new premium without inflation is $\frac{0.829785333504}{0.8}\times 830 = \$860.90$.

10. *An insurance company is calculating the premium for a new line of insurance it started in 2018. The new line of insurance started on 1st May 2018, and half of the policies started at that time. Due to an advertising campaign, the rate of policy purchases in November and December was twice the rate for the months from May to October. The annual premium in 2018 was $600. The total premiums collected in 2018 were $1,200,000 and the total losses were $462,000. Assuming losses are uniformly distributed throughout the year, annual inflation is 5%, and the expense ratio is 0.2, calculate the new premium for policy year 2020. [15 mins]*



The number of policies in force at time t in the year 2018 is

$$f(t) = \begin{cases} 0 & \text{if } t < \frac{4}{12} \\ 0.6t + 0.3 & \text{if } \frac{4}{12} < t < \frac{10}{12} \\ 1.2t - 0.2 & \text{if } \frac{10}{12} < t < 1 \end{cases}$$

The total earned premiums for accident year 2018 are $1200000 \times \left(\frac{1}{2} \times \frac{6}{12} \times \left(\frac{1}{2} + 0.8\right) + \frac{1}{2} \times \frac{2}{12} \times (0.8 + 1)\right) = 570000$
The loss ratio is therefore $\frac{462000}{570000} = 0.810526315789$, so before inflation the premium is should be adjusted by a factor of $\frac{0.810526315789}{0.8} = 1.01315789474$.
Inflation from the start of 2018 to the average accident time in 2018 is given by

$$\frac{\int_{\frac{4}{12}}^{\frac{10}{12}} (0.6t + 0.3)(1.05)^t \, dt + \int_{\frac{10}{12}}^{1} (1.2t - 0.2)(1.05)^t \, dt}{\left(\frac{1}{2} \times \frac{6}{12} \times \left(\frac{1}{2} + 0.8\right) + \frac{1}{2} \times \frac{2}{12} \times (0.8 + 1)\right)}$$

$$= \frac{0.3 \int_{\frac{4}{12}}^{\frac{10}{12}} (1.05)^t \, dt + 0.6 \int_{\frac{4}{12}}^{\frac{10}{12}} t(1.05)^t \, dt + 1.2 \int_{\frac{10}{12}}^{1} t(1.05)^t \, dt - 0.2 \int_{\frac{10}{12}}^{1} (1.05)^t \, dt}{0.475}$$

$$= \frac{0.3(1.05^{\frac{10}{12}} - 1.05^{\frac{4}{12}}) - 0.2(1.05 - 1.05^{\frac{10}{12}}) + 0.6\left(\frac{10}{12}(1.05)^{\frac{10}{12}} - \frac{4}{12}(1.05)^{\frac{4}{12}} - \frac{(1.05)^{\frac{10}{12}} - (1.05)^{\frac{4}{12}}}{\log(1.05)}\right) + 1.2\left(1.05 - \frac{10}{12}1.05^{\frac{10}{12}} - \cdots\right.}{0.475 \log(1.05)}$$

$$= \frac{1.05 - 0.5(1.05)^{\frac{4}{12}} - \frac{1.2(1.05) - 0.6(1.05)^{\frac{10}{12}} - 0.6(1.05)^{\frac{4}{12}}}{\log(1.05)}}{0.475 \log(1.05)}$$

$= 1.03492570259$

Inflation from the start of 2020 to the average accident time in policy year 2020 is given by

$$\begin{aligned}
\int_0^1 t(1.05)^t\,dt + \int_1^2 (2-t)(1.05)^t\,dt &= \int_0^1 t(1.05)^t\,dt + (1.05)\int_0^1 (1-t)(1.05)^t\,dt \\
&= 1.05\int_0^1 (1.05)^t\,dt - 0.05\int_0^1 t(1.05)^t\,dt \\
&= \frac{1.05\times 0.05}{\log(1.05)} - 0.05\left(\frac{1.05}{\log(1.05)} - \frac{0.05}{\log(1.05)^2}\right) \\
&= \left(\frac{0.05}{\log(1.05)}\right)^2 \\
&= 1.05020830855
\end{aligned}$$

The premium is therefore $600 \times 1.01315789474 \times 1.05^2 \times \frac{1.05020830855}{1.03492570259} = \$680.10$.

11. *For a certain line of insurance, the loss amount per claim follows a Pareto distribution with $\alpha = 4$. If the policy has a deductible per loss set at 0.1θ and a policy limit set at 2θ, by how much will the expected payment per loss increase if there is inflation of 5%? [15 mins]*

The expected payment per loss before inflation is

$$\begin{aligned}
\int_{0.1\theta}^{2.1\theta} S(x)\,dx &= \int_{0.1\theta}^{2.1\theta} \frac{\theta^4}{(\theta+x)^4}\,dx \\
&= \theta^4\int_{1.1\theta}^{3.1\theta} u^{-4}\,du \\
&= \theta^4\left[-\frac{u^{-3}}{3}\right]_{1.1\theta}^{3.1\theta} \\
&= \theta^4\left(\frac{(1.1\theta)^{-3}}{3} - \frac{(3.1\theta)^{-3}}{3}\right) \\
&= \frac{\theta}{3}\left((1.1)^{-3} - 3.1^{-3}\right) \\
&= 0.2392492053394\theta
\end{aligned}$$

After inflation, the expected payment per loss is

$$\begin{aligned}
\int_{0.1\theta}^{2.1\theta} S(x)\,dx &= \int_{0.1\theta}^{2.1\theta} \frac{(1.05\theta)^4}{(1.05\theta+x)^4}\,dx \\
&= 1.05^4\theta^4 \int_{1.15\theta}^{3.15\theta} u^{-4}\,du \\
&= \theta^4\left[-\frac{u^{-3}}{3}\right]_{1.15\theta}^{3.15\theta} \\
&= 1.05^4\theta^4\left(\frac{(1.15\theta)^{-3}}{3} - \frac{(3.15\theta)^{-3}}{3}\right) \\
&= \frac{1.05^4\theta}{3}\left((1.15)^{-3} - 3.15^{-3}\right) \\
&= 0.2534420670036\theta
\end{aligned}$$

The increase in expected payment per loss is $\frac{0.253442067036}{0.239249205394} - 1 = 5.93225\%$.

12. *The random variable X has density function given by*

$$f(x) = \frac{15}{4}x(1-x)^2(2-x), 0 \leqslant x \leqslant 2$$

(a) calculate the hazard rate of X. [10 mins]

The distribution function is

$$F(x) = \frac{15}{4}\int 4x^3 - x^4 - 5x^2 + 2xdx = \frac{15}{4}\left(x^4 - \frac{x^5}{5} - \frac{5}{3}x^3 + x^2\right) = \frac{x^2}{4}\left(15x^2 - 3x^3 + 15 - 25x\right)$$

The survival function is therefore

$$\begin{aligned}
S(x) &= \frac{1}{4}\left(4 + 3x^5 + 25x^3 - 15x^4 - 15x^2\right) = \frac{1}{4}(x-2)\left(3x^4 - 9x^3 + 7x^2 - x - 2\right) \\
&= \frac{1}{4}(x-2)^2(3x^3 - 3x^2 + x + 1)
\end{aligned}$$

The hazard rate is therefore

$$\lambda(x) = \frac{f(x)}{S(x)} = \frac{\frac{15}{4}x(1-x)^2(2-x)}{\frac{1}{4}(x-2)^2(3x^3-3x^2+x+1)} = \frac{15x(1-x)^2}{(x-2)(3x^3-3x^2+x+1)}$$

(b) Calculate the kurtosis of X [15 mins]

The raw moments are

$$
\begin{aligned}
\mu &= 3.75\int_0^2 x^2(x-1)^2(x-2)dx = 3.75\int_0^2 (4x^4+2x^2-x^5-5x^3)dx \\
&= 3.75\left(0.8\times 2^5+\frac{2}{3}\times 2^3-\frac{2^6}{6}-1.25\times 2^4\right) = 96+20-40-75 = 1 \\
\mu_2' &= 3.75\left(\frac{4}{6}\times 2^6+\frac{2}{4}\times 2^4-\frac{2^7}{7}-2^5\right) = 160+30-\frac{480}{7}-120 = \frac{10}{7} \\
\mu_3' &= 3.75\left(\frac{4}{7}\times 2^7+\frac{2}{5}\times 2^5-\frac{2^8}{8}-\frac{5}{6}\times 2^6\right) = \frac{1920}{7}+48-120-200 = \frac{16}{7} \\
\mu_4' &= 3.75\left(\frac{4}{8}\times 2^8+\frac{2}{6}\times 2^6-\frac{2^9}{9}-\frac{5}{7}\times 2^7\right) = 480+80-\frac{640}{3}-\frac{2400}{7} = \frac{80}{21}
\end{aligned}
$$

The centralised moments are

$$
\begin{aligned}
\mu_2 &= \frac{10}{7}-1^2 = \frac{3}{7} \\
\mu_3 &= \frac{16}{7}-3\times\frac{10}{7}\times 1+2\times 1^3 = 0 \\
\mu_4 &= \frac{80}{21}-4\times\frac{16}{7}\times 1+6\times\frac{10}{7}\times 1^2-3\times 1^4 = \frac{5}{21}
\end{aligned}
$$

The Kurtosis is therefore

$$
\frac{\left(\frac{5}{21}\right)}{\left(\frac{3}{7}\right)^2} = \frac{35}{27}
$$

13. *Losses follow a Pareto distribution with $\alpha = 3$. How large can θ be if the insurance company wants to limit its Value at Risk at the 95% level to \$15,000? [10 mins]*

Recall that Value at Risk is the 95th percentile, so we need to solve

$$
\begin{aligned}
\left(\frac{\theta}{15000+\theta}\right)^3 &= 0.05 \\
\frac{\theta}{15000+\theta} &= 0.05^{\frac{1}{3}} \\
\frac{15000+\theta}{\theta} &= 20^{\frac{1}{3}} \\
\frac{15000}{\theta} &= 20^{\frac{1}{3}}-1 \\
\theta &= \frac{15000}{20^{\frac{1}{3}}-1} = 8749.33
\end{aligned}
$$

14. *Calculate the moment generating function of a sum of 5 independent beta random variables with parameters 3 and 2. [10 mins]*

$$\begin{aligned}\mathbb{E}(e^{tX}) &= 12\int_0^1 x^2(1-x)e^{tx}dx = 12\left(\left[\frac{x^2(1-x)e^{tx}}{t}\right]_0^1 - \int_0^1 \frac{(2x-3x^2)e^{tx}}{t}dx\right)\\ &= 12\left(-\left[\frac{(2x-3x^2)e^{tx}}{t^2}\right]_0^1 + \int_0^1 \frac{(2-6x)e^{tx}}{t^2}dx\right)\\ &= 12\left(\frac{e^t}{t^2} + \left[\frac{(2-6x)e^{tx}}{t^3}\right]_0^1 + \int_0^1 \frac{6e^{tx}}{t^3}dx\right) = 12\left(\frac{e^t}{t^2} - \frac{2}{t^3} - \frac{4e^t}{t^3} + \frac{6e^t}{t^4} - \frac{6}{t^4}\right)\\ &= \frac{12(t^2-4t+6)e^t - 24(t+3)}{t^4}\end{aligned}$$

The moment generating function of a sum of 5 beta random variables is therefore

$$\left(\frac{12(t^2-4t+6)e^t - 24(t+3)}{t^4}\right)^5$$

15. *Which distribution has a heavier tail: a gamma distribution with $\alpha = 4$ and $\theta = 400$, or a Weibull distribution with $\tau = 4$ and $\theta = 400$? [Use any reasonable method for comparing tail-weight.] [5 mins]*

The easiest method is to look at the ratio of their density functions. This is

$$\frac{x^3 e^{-\frac{x}{400}} \times x}{6 \times 4\left(\frac{x}{400}\right)^4 e^{-\left(\frac{x}{400}\right)^4}} = \frac{e^{\left(\frac{x}{400}\right)^4 - \frac{x}{400}}}{24}$$

This clearly tends to ∞ as $x \to \infty$, so the gamma distribution has a heavier tail.

Alternative solution:

The nth moment of the gamma distribution is $\mu'_n = 400^n \frac{\Gamma(n+4)}{\Gamma(4)}$, while the nth moment of the Weibull distribution is $400^n \Gamma\left(1 + \frac{n}{4}\right)$. As n gets large, the moments of the gamma distribution are larger than the moments of the Weibull distribution, so the gamma has the heavier tail.

[Other approaches include comparing hazard rates, or survival functions, but these are harder to compute for the gamma distribution.]

16. *Recall that desirable coherence properties for measures of risk are:*

- *Subadditivity*
- *Monotonicity*

- *Positive homogeneity*
- *Translation invariance*

Which properties are satisfied by the risk measure given by the measure $r(X) = \mu + \pi_{0.9}$ (the mean plus the 90th percentile)? [10 mins]

The mean satisfies all of the properties, while the 90th percentile satisfies all except subadditivity, so the sum of risk measures must also satisfy this property. The only thing remaining is to check whether subadditivity is satisfied. Let X be a Bernoulli random variable with $p = 0.09$. The 90th percentile of X is therefore 0, and the mean is 0.09. Let Y be another Bernoulli variable mutually exclusive with X (That is, we can't have $X = Y = 1$) with $p = 0.02$. Then $X+Y$ is a Bernoulli random variable with $p = 0.11$, so we have $r(X+Y) = 0.11+1 = 1.11$, while $r(X) = 0.09$ and $r(Y) = 0.02$, so the measure is not subadditive.

17. *Calculate the TVaR of a gamma distribution with $\alpha = 3$ and $\theta = 2000$ at the 0.99 level. [The VaR at the 0.99 level is 16,811.894] [15 mins]*

This is the conditional expectation given that the value is above the 99th percentile. Let π be the 99th percentile of a gamma distribution with $\alpha = 3$ and $\theta = 1$. Then the TVaR is $2000\frac{\int_\pi^\infty x^3e^{-x}\,dx}{\int_\pi^\infty x^2e^{-x}\,dx}$. Integrating by parts, we get

$$\int_\pi^\infty x^3e^{-x}dx = [-x^3e^{-x}]_\pi^\infty + \int_\pi^\infty 3x^2e^{-x}dx$$

and $\int_\pi^\infty x^3e^{-x}dx = [-x^2e^{-x}]_\pi^\infty + \int_\pi^\infty 2xe^{-x}dx = \pi^2e^{-\pi} + 2\pi e^{-\pi} + 2e^{-\pi}$. The TVaR is therefore $2000\left(\frac{\pi^3+3\pi^2+6\pi+6}{\pi^2+2\pi+2}\right)$ Furthermore, π is the 99th percentile, meaning that $\int_\pi^\infty x^2e^{-x}dx = 0.01\Gamma(3) = 0.02$. We therefore have that $(\pi^2+2\pi+2)e^{-\pi} = 0.02$ and the TVaR is

$$2000\left(\frac{\pi^3}{0.02e^\pi}+3\right)$$

We are given that $\pi = \frac{16811.894}{2000} = 8.405947$, so the TVaR is

$$2000\left(\frac{8.405947^3}{0.02e^{8.405947}}+3\right) = \$19,277.11$$

18. *Claims follow a Pareto distribution with $\alpha = 4$. There is a policy limit which is currently exceeded by 0.16% of claims. There is uniform inflation of 8% per year on claim amounts. What proportion of claims will exceed the policy limit in 4 years time? [The policy limit does not change in these 4 years.] [5 mins]*

In 4 years time, the loss distribution will be Pareto with $\alpha = 4$ and $\theta = (1.08)^4\theta_0$ (where θ_0 is the current parameter value). We have that the policy limit l is set so that $\left(\frac{\theta_0}{\theta_0+l}\right)^4 = 0.0016$. This gives $\frac{\theta_0+l}{\theta_0} = 5$, so $l = 4\theta_0$. In 4 years time, we will have $\theta = (1.08)^4\theta_0$, so the probability of exceeding the policy limit will be $\left(\frac{(1.08)^4\theta_0}{(1.08)^4\theta_0+4\theta_0}\right)^4 = \left(\frac{(1.08)^4}{(1.08)^4+4}\right)^4 = 0.004149182$, so 0.415% of claims will exceed the limit.

19. *You observe the following sample of insurance losses:*

1.6 3.6 3.8 4.2 5.6

Using a Kernel density model with Gaussian (normal) kernel with standard deviation 1.2, estimate the probability that a loss exceeds 5.5. [5 mins]

This probability is given by

$$\begin{aligned}
&1-\frac{1}{5}\left(\Phi\left(\frac{5.5-1.6}{1.2}\right)+\Phi\left(\frac{5.5-3.6}{1.2}\right)+\Phi\left(\frac{5.5-3.8}{1.2}\right)+\Phi\left(\frac{5.5-4.2}{1.2}\right)+\Phi\left(\frac{5.5-5.6}{1.2}\right)\right)\\
&=1-\frac{1}{5}\left(\Phi(3.25)+\Phi(1.58)+\Phi(1.42)+\Phi(1.08)+\Phi(-0.08)\right)\\
&=1-\frac{1}{5}(0.9994+0.9429+0.9222+0.8599+0.4681)=1-\frac{4.1925}{5}=1-0.8385=0.1615
\end{aligned}$$

20. *You observe the following sample of insurance losses:*

    1.6 3.6 3.8 4.2 5.6

    Using a Kernel density model with triangular kernel with bandwidth 2, estimate the probability that a loss exceeds 5.5. A triangular kernel with bandwidth b centred at x_0 is given by the density function

$$f(x)=\begin{cases}\frac{x+b-x_0}{b^2} & \text{if } x_0-b<x<x_0\\ \frac{x_0+b-x}{b^2} & \text{if } x_0<x<x_0+b\\ 0 & \text{otherwise}\end{cases}$$

[5 mins]

For the triangular distribution centred at x_0, the probability of exceeding x_0+2-a for $0<a<2$ is $\frac{a^2}{8}$, and the probability of exceedint x_0-2+a for $0<a<2$ is $1-\frac{a^2}{8}$. Therefore, the probability of exceeding 5.5 is given by

$$\frac{1}{5}\left(0+\frac{0.01}{8}+\frac{0.09}{8}+\frac{0.49}{8}+1-\frac{3.61}{8}\right)=\frac{4.98}{40}=0.1245$$