ACSC/STAT 3703, Actuarial Models I

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Toby Kenney

Homework Sheet 3

Model Solutions

Basic Questions

1. *A distribution has survival function $S(x) = \frac{3}{(2+x)^2} + \frac{16}{(4+x)^3}$ for $x \geqslant 0$. Calculate its hazard-rate.*

The density function is

$$f(x) = -\frac{d}{dx}S(x) = -\left(-\frac{6}{(x+2)^3} - \frac{48}{(x+4)^4}\right) = \frac{6}{(x+2)^3} + \frac{48}{(x+4)^4}$$

so the hazard rate is

$$\lambda(x) = \frac{f(x)}{S(x)} = \frac{\frac{6}{(x+2)^3} + \frac{48}{(x+4)^4}}{\frac{3}{(2+x)^2} + \frac{16}{(4+x)^3}} = \frac{6(x+4)^4 + 48(x+2)^3}{3(x+4)^4(x+2) + 16(x+2)^3(x+4)} = 6\frac{x^4 + 32x^3 + 192x^2 + 448x + 384}{(x+2)(x+4)(3x^3 + 52x^2 + 208x + 256)}$$

2. *A continuous random variable has moment generating function given by $M(t) = \frac{1}{(1-2t)^3(1-5t)^2}$ for $t < 0.2$. Calculate its coefficient of variation.*

We calculate

$$\begin{aligned}
M'(t) &= \frac{6(1-5t) + 10(1-2t)}{(1-2t)^4(1-5t)^3} \\
&= \frac{16-50t}{(1-2t)^4(1-5t)^3} \\
M''(t) &= \frac{(8(1-5t) + 15(1-2t))(16-50t) - 50(1-2t)(1-5t)}{(1-2t)^5(1-5t)^4} \\
M''(t) &= \frac{318 - 1520t + 3000t^2}{(1-2t)^5(1-5t)^4} \\
M'(0) &= 16 \\
M''(0) &= 318
\end{aligned}$$

Thus $\mathbb{E}(X) = 16$ and $\mathbb{E}(X^2) = 318$. This gives $\text{Var}(X) = 318 - 16^2 = 62$. Thus, the coefficient of variation is $\frac{\sqrt{62}}{16} = 0.492125492126$.

3. *Calculate the mean excess loss function for a distribution with survival function given by* $S(x) = \frac{2}{(x+1)^3} - \frac{32}{(x+2)^5}$ *for* $x \geqslant 0$.

   The mean excess loss function is given by

$$\begin{aligned}
\mathbb{E}((X-d)_+) &= \int_d^\infty S(x)\,dx \\
&= \int_d^\infty \left(\frac{2}{(x+1)^3} - \frac{32}{(x+2)^5}\right) dx \\
&= \left[-\frac{1}{(x+1)^2} + \frac{8}{(x+2)^4}\right]_d^\infty \\
&= \frac{1}{(d+1)^2} - \frac{8}{(d+2)^4}
\end{aligned}$$

4. *Calculate the probability generating function of a discrete distribution with p.m.f. given by*

$$f(x) = \frac{x^2}{6 \times 2^x}$$

   for $n \geqslant 0$.

   [We can show this is a probability mass function as follows:

   We need to show

$$\sum_{n=0}^\infty \frac{n^2}{2^n} = 6$$

   To do this, we combine the n*th and* $n+1$*th terms in the series*

$$\sum_{n=0}^\infty \frac{n^2}{2^n} = \frac{1}{2}\sum_{n=0}^\infty \left(\frac{n^2}{2^n} + \frac{(n+1)^2}{2^{n+1}}\right) + \frac{1}{2}\frac{0^2}{2^0}$$

   The term in the sum can be rearranged to give

$$\sum_{n=0}^\infty \frac{n^2}{2^n} = \frac{1}{2}\sum_{n=0}^\infty \frac{\frac{3}{2}n^2 + n + \frac{1}{2}}{2^n} = \frac{3}{4}\sum_{n=0}^\infty \frac{n^2}{2^n} + \frac{1}{2}\sum_{n=0}^\infty \frac{n}{2^n} + \frac{1}{4}\sum_{n=0}^\infty \frac{1}{2^n}$$

   A similar argument gives

$$\sum_{n=0}^\infty \frac{n}{2^n} = \frac{1}{2}\sum_{n=0}^\infty \left(\frac{n}{2^n} + \frac{n+1}{2^{n+1}}\right) + \frac{1}{2}\frac{0}{2^0} = \frac{1}{2}\sum_{n=0}^\infty \frac{\frac{3}{2}n + \frac{1}{2}}{2^n} = \frac{3}{4}\sum_{n=0}^\infty \frac{n}{2^n} + \frac{1}{4}\sum_{n=0}^\infty \frac{1}{2^n}$$

   We know that $\sum_{n=0}^\infty \frac{1}{2^n} = 2$, *so we can solve these to get*

$$\sum_{n=0}^\infty \frac{n}{2^n} = 2$$

and

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} = 6$$

You may need to do a similar derivation to get the probability generating function.]

The probability generating function is given by

$$P(z) = \mathbb{E}(z^X) = \sum_{n=0}^{\infty} f(x) z^x = \frac{1}{6} \sum_{n=0}^{\infty} \frac{x^2 z^x}{2^x} = \frac{1}{6} \sum_{n=0}^{\infty} x^2 \left(\frac{z}{2}\right)^x$$

Similarly to the argument that $\sum_{n=0}^{\infty} \frac{n^2}{2^n} = 6$, we derive

$$\begin{aligned}
\sum_{n=0}^{\infty} na^n &= \frac{1}{2} \sum_{n=0}^{\infty} na^n + (n+1)a^{n+1} \\
&= \frac{1+a}{2} \sum_{n=0}^{\infty} na^n + \frac{a}{2} \sum_{n=0}^{\infty} a^n \\
&= \frac{1+a}{2} \sum_{n=0}^{\infty} na^n + \frac{a}{2(1-a)} \\
\sum_{n=0}^{\infty} na^n &= \frac{\left(\frac{a}{2(1-a)}\right)}{1 - \frac{1+a}{2}} \\
&= \frac{a}{(1-a)^2} \\
\sum_{n=0}^{\infty} n^2 a^n &= \frac{1}{2} \sum_{n=0}^{\infty} n^2 a^n + (n+1)^2 a^{n+1} \\
&= \frac{1+a}{2} \sum_{n=0}^{\infty} n^2 a^n + a \sum_{n=0}^{\infty} na^n + \frac{a}{2} \sum_{n=0}^{\infty} a^n \\
&= \frac{1+a}{2} \sum_{n=0}^{\infty} n^2 a^n + \frac{a^2}{(1-a)^2} + \frac{a}{2(1-a)} \\
\sum_{n=0}^{\infty} n^2 a^n &= \frac{2}{1-a} \left(\frac{a^2}{(1-a)^2} + \frac{a}{2(1-a)} \right) \\
&= \frac{2a^2 + a(1-a)}{(1-a)^3} \\
&= \frac{a^2 + a}{(1-a)^3}
\end{aligned}$$

Applying this to $a = \frac{z}{2}$ gives

$$P(z) = \frac{1}{6}\left(\frac{2z^2+4z}{(2-z)^3}\right)$$

Standard Questions

5. *The total cost of handling a claim is $X+Y$ where X is a discrete non-negative random variable with probability generating function $P_X(z) = e^{-4(1-(2.6-1.6z)^{-2})}$ and Y is a continuous non-negative random variable with moment generating function $M_Y(t) = \left(0.6 + \frac{0.4}{(1-t)^2}\right)^3$. X and Y are independant. What is the moment generating function of the total cost?*

   Recall that

   $$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \mathbb{E}\left(\left(e^t\right)^X\right) = P_X\left(e^t\right) = e^{-4(1-(2.6-1.6e^t)^{-2})}$$

   Therefore,

   $$M_{X+Y}(t) = \left(0.6 + \frac{0.4}{(1-t)^2}\right)^3 e^{-4(1-(2.6-1.6e^t)^{-2})}$$

6. *An insurance company is trying to fit an inverse Pareto distribution to its claims data. The survival function for this distribution is given by*

   $$S(x) = 1 - \frac{x^\tau}{(x+\theta)^\tau}$$

   The insurance company wants to select α and θ so that the the 5th percentile and the 95th percentile match the observed values of 458 and 86,322 respectively. Which of the following values should they choose for τ and what should be the corresponding value be for θ?

   (i) 1.467008

   (ii) 1.882693

   (iii) 2.898321

   (iv) 4.405930

   We need to solve the equations

   $$1 - \frac{458^\tau}{(\theta+458)^\tau} = 0.95$$
   $$1 - \frac{86322^\tau}{(\theta+86322)^\tau} = 0.05$$

We rearrange these to get

$$\theta = 458\left(0.05^{-\frac{1}{\tau}} - 1\right) = 86322\left(0.95^{-\frac{1}{\tau}} - 1\right)$$

We try the given values of τ and get the following:

τ	$458\left(0.05^{-\frac{1}{\tau}} - 1\right)$	$86322\left(0.95^{-\frac{1}{\tau}} - 1\right)$
(i) 1.467008	3071.59591188	3071.59665137
(ii) 1.882693	1790.60318345	2384.14195131
(iii) 2.898321	829.527172559	1541.28962116
(iv) 4.405930	445.974421654	1010.82261104

So the value of τ is (i) $\tau = 1.467008$ and the corresponding value for θ is $\theta = 3071.596$.

Bonus Question

7. *For a particular infectious disease, the number of distinct uninfected people, N, infected by a single infected person has a distribution with probability generating function $P(z) = 1 - \sqrt{\frac{1-z}{2}}$.*

A pandemic begins with a single infected person, and the numbers of people infected by different people are independent.

(a) What is the probability that the pandemic ever dies out (i.e. that only a finite number of total infections happen)?

Let this probability be z, and let N_1 be the number of people infected by the original infected person. For each future infected person, we can consider all the people infected directly or indirectly from this individual, and ask whether this number is finite. Because all the distributions are identical, the probability that the number is finite is z for any individual. The total number of people infected is only finite if for every infected person, the number of people infected directly or indirectly from this person is finite. For the N_1 individuals infected by the first individual, these events are independent, and all have probability z. If the numbers for all N_1 of these individuals are finite, then the total number is finite. For a fixed value $N_1 = n$, the probability of this is therefore z^n. Thus, we deduce

$$z = \mathbb{E}\left(z^{N_1}\right) = P_{N_1}(z)$$

We can therefore find the solution to this.

$$\begin{aligned} z &= 1 - \sqrt{\frac{1-z}{2}} \\ \sqrt{\frac{1-z}{2}} &= 1-z \\ \frac{1-z}{2} &= (1-z)^2 \\ 1-z &= \frac{1}{2} \\ z &= \frac{1}{2} \end{aligned}$$

(b) What is the "probability generating function" for the total number of people infected? [Technically, since there is non-zero probability that this number is infinite, it is not a probability generating function.]

Let T_i be the total number of people infected by the ith infected person. The total number of people infected is $T_1 = 1 + T_2 + \cdots + T_{N_1+1}$, and since $T_2, \ldots, T_{N_1+1}$ are i.i.d., the probability generating function for T_1 is

$$P_{T_1}(z) = \mathbb{E}_{N_1}(zP_{T_2}(z)\cdots P_{T_{N_1+1}}(z)) = \mathbb{E}_{N_1}\left(zP_{T_1}(z)^{N_1}\right) = zP_{N_1}(P_{T_1}(z))$$

Substituting $P_{N_1}(z) = 1 - \sqrt{\frac{1-z}{2}}$, we get

$$\begin{aligned} P_{T_1}(z) &= z\left(1 - \sqrt{\frac{1-P_{T_1}(z)}{2}}\right) \\ \sqrt{\frac{1-P_{T_1}(z)}{2}} &= 1 - \frac{P_{T_1}(z)}{z} \\ 1 - P_{T_1}(z) &= 2\left(1 - 2\frac{P_{T_1}(z)}{z} + \frac{P_{T_1}(z)^2}{z^2}\right) \\ P_{T_1}(z)^2 - \left(2z - \frac{z^2}{2}\right)P_{T_1}(z) + \frac{z^2}{2} &= 0 \\ P_{T_1}(z) &= \frac{\left(2z - \frac{z^2}{2}\right) \pm \sqrt{\left(2z - \frac{z^2}{2}\right)^2 - 2z^2}}{2} \\ &= \frac{\left(2z - \frac{z^2}{2}\right) \pm z\sqrt{2 - 2z + \frac{z^2}{4}}}{2} \end{aligned}$$

We also have $1 - \frac{P_{T_1}(z)}{z} \geqslant 0$, so we must take the smaller solution

$$P_{T_1}(z) = \frac{\left(2z - \frac{z^2}{2}\right) - z\sqrt{2 - 2z + \frac{z^2}{4}}}{2} = z\left(1 - \frac{z}{4} + \sqrt{\frac{1}{2} - \frac{z}{2} + \frac{z^2}{16}}\right)$$

[We can also see that the smaller solution is the correct one by noting that from (a), the probability that T_1 is finite is $\frac{1}{2}$, so $P_{T_1}(1) = \frac{1}{2}$, which only works for the smaller root of the quadratic equation.]