ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 4

Model Solutions

Basic Questions

1. *A distribution has survival function*

$$S(x) = \begin{cases} 1 & \textit{if } x \leqslant 1 \\ x^{-1}2^{-n} & \textit{if } 2^n \leqslant x \leqslant 2^{n+1} - \frac{1}{n} \\ \frac{4^n(n-1)(1+(n-1)(2^n-x))+2^n((n-1)(2^n-x)-1)}{16^n(n-1)-8^n} & \textit{if } 2^n - \frac{1}{n-1} \leqslant x \leqslant 2^n \end{cases}$$

where $n \geqslant 0$. How does the tail weight of this distribution compare to that of a Weibull distribution with $\tau = 0.5$ and $\theta = 1$, when tail-weight is assessed by

(a) Asymptotic behaviour of hazard rate.

We differentiate $S(x)$ to get

$$f(x) = \begin{cases} x^{-2}2^{-n} & \texttt{if } 2^n \leqslant x \leqslant 2^{n+1} - \frac{1}{n} \\ (n-1)\frac{4^n(n-1)+2^n}{16^n(n-1)-8^n} & \texttt{if } 2^n - \frac{1}{n-1} \leqslant x \leqslant 2^n \end{cases}$$

so

$$\lambda(x) = \frac{f(x)}{S(x)} = \begin{cases} x^{-1} & \texttt{if } 2^n \leqslant x \leqslant 2^{n+1} - \frac{1}{n} \\ \frac{4^n(n-1)^2+2^n(n-1)}{4^n(n-1)(1+(n-1)(2^n-x))+2^n((n-1)(2^n-x)-1)} & \texttt{if } 2^n - \frac{1}{n-1} \leqslant x \leqslant 2^n \end{cases}$$

We see that $\lambda(x)$ does not converge as $x \to \infty$.

For the Weibull distribution, we have $S(x) = e^{-\sqrt{x}}$ and $f(x) = 0.5x^{-0.5}e^{-\sqrt{x}}$, so $\lambda(x) = 0.5x^{-0.5} \to 0$. Thus, the Weibull distribution would appear to have a heavier tail.

(b) Existence of moments.

The kth moment of the Weibull distribution is

$$\begin{aligned} \int_0^\infty 0.5x^{k-0.5}e^{-\sqrt{x}}\,dx &= \int_0^\infty u^{2k}e^{-u} \\ &= \Gamma(2k) \end{aligned}$$

where we have used the substitution $x = u^2$. Thus, all moments exist. For the distribution given with

$$S(x) = \begin{cases} 1 & \text{if } x \leqslant 1 \\ x^{-1}2^{-n} & \texttt{if } 2^n \leqslant x \leqslant 2^{n+1} - \frac{1}{n} \\ \frac{4^n(n-1)(1+(n-1)(2^n-x))+2^n((n-1)(2^n-x)-1)}{16^n(n-1)-8^n} & \texttt{if } 2^n - \frac{1}{n-1} \leqslant x \leqslant 2^n \end{cases}$$

we have $x^{-2} < S(x) < 2x^{-2}$ for all x, so the mean of the distribution exists, but the second moment does not. Therefore this distribution has the heavier tail.

2. *Which coherence properties are satisfied by the following measure of risk?*

$$\rho(X) = \sup_x xP(X > x)$$

Give a proof or a counterexample for each property.

Sub-additivity This does not hold. Let X be uniform on $[0, 1]$, and let $Y = 1 - X$. We have that $\rho(X) = \sup_{x\in[0,1]} x(1 - x) = \frac{1}{4}$ and similarly, $\rho(Y) = \frac{1}{4}$, but $\rho(X + Y) = 1$, since $X + Y = 1$ with probability 1.

Monotonicity If $X < Y$, then for any x, we have $P(X > x) < P(Y > x)$, so

$$\rho(X) < \rho(Y)$$

Positive homogeneity We want to show that for any $a > 0$, $\rho(aX) = a\rho(X)$. It is straightforward to see that for any x, $P(aX > x) = P\left(X > \frac{x}{a}\right)$ so in particular $xP(aX > x) = a\frac{x}{a}P\left(X > \frac{x}{a}\right) \leqslant a\sup_y yP(X > y) = a\rho(a)$. Conversely, $xP(X > x) = \frac{ax}{a}P(aX > ax) \leqslant \frac{\rho(aX)}{a}$. Therefore $\rho(aX) = a\rho(X)$.

Translation invariance This is not true. Let X follow a uniform distribution on $[0, 1]$. Then $\rho(X) = \sup_{x\in[0,1]} x(1-x) = \frac{1}{4}$, and $\rho(X+1) = \sup_{x\in[0,1]}(x + 1)(1 - x) = 1 \neq 1 + \frac{1}{4}$.

3. *Calculate the TVaR at the 95% level of a distribution with survival function $S_X(x) = xe^{-x^2}$ for $x > 1$. [You may need to use numerical methods to find the VaR.]*

Let a be the 95th percentile of the distribution. By definition, this is the solution to $ae^{-a^2} = 0.05$, which numerically is $a = 1.9084020980$.

The TVaR is therefore

$$\begin{aligned}
a+\int_a^\infty S(x)\,dx &= a+\int_a^\infty xe^{-x^2}\,dx\\
&= a+\left[-\frac{1}{2}e^{-x^2}\right]_a^\infty\\
&= a+\frac{e^{-a^2}}{2}\\
&= a+\frac{0.025}{a}\\
&= 1.92150206264
\end{aligned}$$

4. *Which of the following density functions with parameters α, β and γ are scale distributions? Which have scale parameters?*

   (i) $f(x) = Cx^{1-\alpha}(x+1)^{1-\beta}\gamma^{\alpha+\beta}$

   (ii) $f(x) = Cx^{\alpha-1}\gamma^{-\alpha}\gamma^{\frac{x}{\gamma}}e^{-\frac{x\log(x)}{\gamma}}$

   (iii) $f(x) = C\alpha^2\beta^3\gamma^{-1}(x+\alpha)^{-2}(x+\beta)^{-3}e^{-\frac{x}{\gamma}}$

   (i) is not a scale distribution because the constant 1 is not affected by parameters, so for example

   $$f_{2X}(x) = 2^{-1}f_X\left(\frac{x}{2}\right) = 2Cx^{1-\alpha}2^{\alpha-1}(x+2)^{1-\beta}2^{\beta-1}\gamma^{\alpha+\beta}$$

   which is not from this parametric distribution family.

   (ii) is a scale distribution, and γ is a scale parameter, since

   $$f_{cX}(x) = c^{-1}f_X\left(\frac{x}{c}\right) = c^{-1}Cx^{\alpha-1}c^{1-\alpha}\gamma^{-\alpha}\gamma^{\frac{x}{c\gamma}}e^{-\frac{x(\log(x)-\log(c))}{c\gamma}} = Cx^{\alpha-1}(c\gamma)^{-\alpha}(c\gamma)^{\frac{x}{c\gamma}}e^{-\frac{x\log(x)}{c\gamma}}$$

   which is the same distribution with γ replaced by $c\gamma$.

   (iii) is a scale distribution, but with no scale parameter. We have

   $$\begin{aligned}
   f_{cX}(x) &= c^{-1}f_X\left(\frac{x}{c}\right)\\
   &= Cc^{-1}\alpha^2\beta^3\gamma^{-1}\left(\frac{x}{c}+\frac{c\alpha}{c}\right)^{-2}\left(\frac{x}{c}+\frac{c\beta}{c}\right)^{-3}e^{-\frac{x}{c\gamma}}\\
   &= C(c\alpha)^2(c\beta)^3(c\gamma)^{-1}(x+c\alpha)^{-2}(x+c\beta)^{-3}e^{-\frac{x}{c\gamma}}
   \end{aligned}$$

   which is the same distribution but with α, *beta* and γ replaced by $c\alpha$, $c\beta$ and $c\gamma$.

5. *An insurance company observes the following sample of claims (in thousands):*

0.5 1.4 1.6 2.1 2.8 3.9 5.6

They use a kernel density model with the following kernel

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the TVaR at the 95% level of the fitted distribution?

There are 7 sample points, and the kernels about the smallest 6 all have support contained in $(-\infty, 4.9]$, so $S(5.6) = \frac{1}{14}$, and for $5.6 < x < 6.6$, we have

$$S(x) = \frac{1}{7}\int_x^{6.6} (1 - x)\,dx = \frac{1}{7}\frac{(6.6 - x)^2}{2}$$

Therefore, the VaR at the 95% level is the solution to

$$\begin{aligned} \frac{1}{7}\frac{(6.6 - x)^2}{2} &= 0.05 \\ (6.6 - x)^2 &= 0.7 \\ x &= 6.6 - \sqrt{0.7} \\ &= 5.76333997347 \end{aligned}$$

The TVaR is therefore

$$\begin{aligned} 5.76333997347 + \frac{1}{0.05}\int_{5.76333997347}^{6.6} \frac{(6.6 - x)^2}{14}\,dx &= 5.76333997347 - 20\left[\frac{(6.6 - x)^3}{42}\right]_{5.76333997347}^{6.6} \\ &= 5.76333997347 + 20\frac{(6.6 - 5.76333997347)^3}{42} \\ &= 6.04222664898 \end{aligned}$$

Standard Questions

6. *A Pareto distribution with α and $\theta = 1$ has mean $\frac{1}{\alpha-1}$ and variance $\frac{\alpha}{(\alpha-1)^2(\alpha-2)}$. You can simulate n random variables following this Pareto distribution with the command*

```
sim=runif(n)^(-1/alpha)-1
```

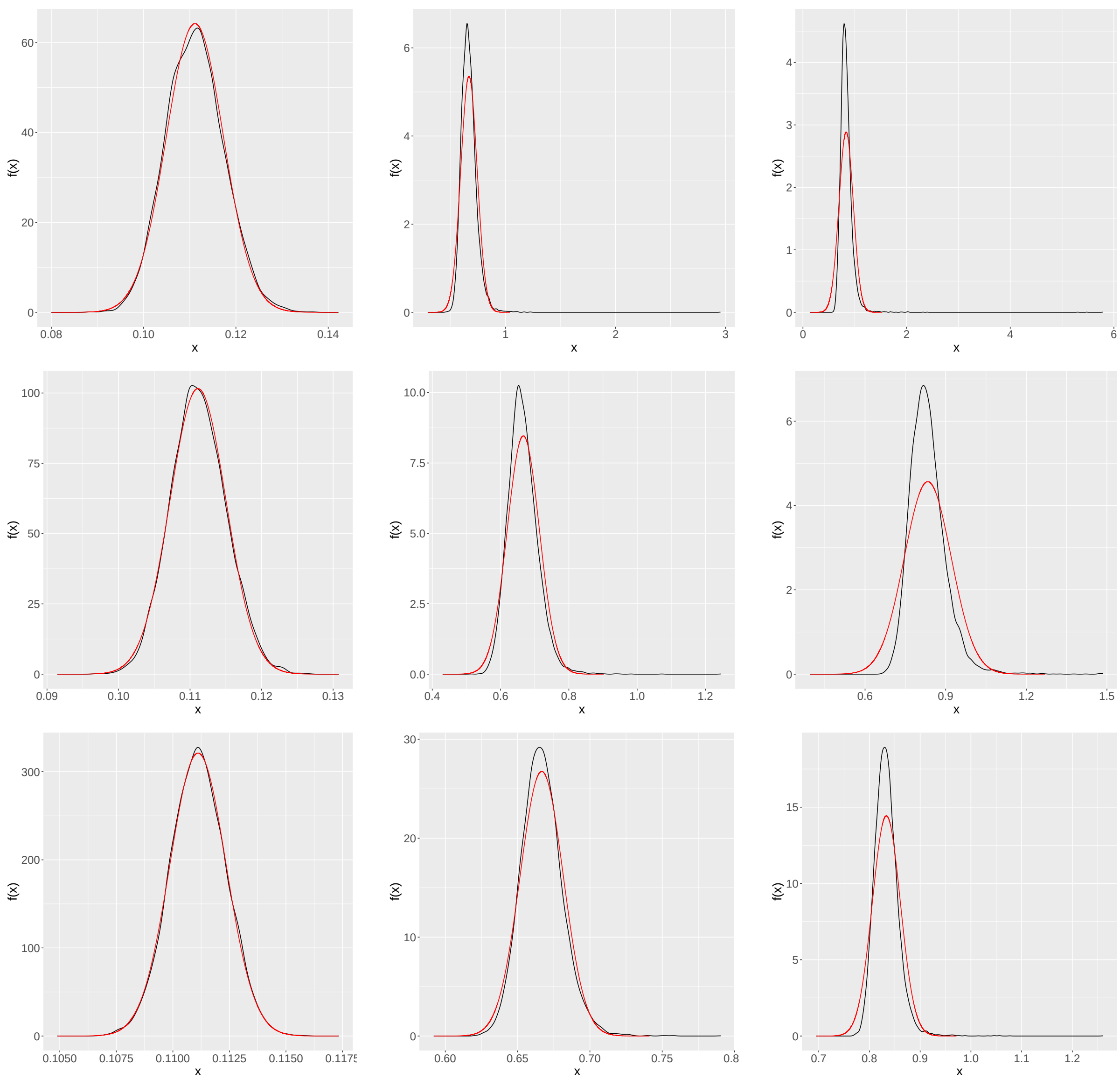
[This is simulating a uniform distribution then transforming the result.]

Based on the central limit theorem, if we take the average of a sample of n Pareto random variables, this should approximately follow a normal distribution with mean $\frac{1}{\alpha-1}$ and variance $\frac{\alpha}{n(\alpha-1)^2(\alpha-2)}$. Plot the distribution *of this sample average for* $\alpha = 10$*,* $\alpha = 2.5$ *and* $\alpha = 2.2$*, for sample sizes 400, 1000, and 10000, and compare it with the normal distribution.*

We run the simulations using the following code

```
library(ggplot2)

paretoCLTplot<-function(alpha,n,nsamp){
### alpha is the pareto parameter
### n is the sample size
### m is the number of samples

    samp<-runif(n*nsamp)^(-1/alpha)-1
    ## simulate Pareto Random Varibles

    samples<-matrix(samp,n,nsamp)
    means<-colMeans(samples)
    ## arranging into a matrix and using the column means function is
    ## an efficient way to calculate the sample means. You could also
    ## use a loop.

    dm<-1/(alpha-1)
    dv<-alpha/(alpha-1)^2/(alpha-2)

    x<-seq_len(100000)*0.0001*sqrt(dv/n)+dm-5*sqrt(dv/n)
    ## x covers 5 standard deviations either side of the mean

    return(
        ggplot(data=data.frame(x=means),mapping=aes(x=x))+
        geom_density()+
        geom_line(data=data.frame(x=x,y=dnorm(x-dm,sd=sqrt(dv/n))),
                  mapping=aes(x=x,y=y),
                  colour="red")+
        scale_y_continuous(name="f(x)")+
        theme(axis.title=element_text(size=18),
            axis.text=element_text(size=16),
            plot.title=element_text(size=18,hjust=0.5))
    )

}


for(alpha in c(10,2.5,2.2)){
    for(ss in c(400,1000,10000)){
        pdf(paste("alpha",alpha,"ssize",ss,".pdf",sep=""))
        print(paretoCLTplot(alpha,ss,10000))
        dev.off()
    }
}
```

Sample size	$\alpha = 10$	$\alpha = 2.5$	$\alpha = 2.2$
400			
1000			
10000			



7. *An insurance company uses a kernel density model for losses, using a gamma kernel with a fixed value of $\alpha = 10$ and $\theta = \frac{x}{10}$ for each observed sample x. The largest eight losses in the sample were*

\$ 5,031,900
\$ 2,528,000
\$ 2,200,600
\$ 1,511,800
\$ 1,273,400
\$ 1,152,400
\$ 947,800
\$ 789,400

Using this model, the Var at the 95% level is \$2,098,300 and the TVaR at the same level is \$2,180,610. How many claims were in the sample?

[You may find it helpful to use the `pgamma` function in `R` to find the distribution function of a gamma distribution.]

Let n be the number of claims in the sample. The largest four claims are as given, and the remaining claims are all at most \$1,511,800. We have that the probability that the kernel density model is less than \$2,098,300 is 0.95.

Let $a = 2098300$. From the TVaR, we have

$$\frac{1}{n}\sum_{i=1}^{n}\int_a^\infty (x-a)\frac{x^9e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)}\,dx = 2180610 - 2098300 = 82310$$

We also have

$$\begin{aligned}\int_a^\infty (x-a)\frac{x^9e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)}\,dx &= \int_a^\infty \frac{x^{10}e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)}\,dx - a\int_a^\infty \frac{x^9e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)}\,dx \\ &= 10\theta_i\int_a^\infty \frac{x^{10}e^{-\frac{x}{\theta_i}}}{(\theta_i)^{11}\Gamma(11)}\,dx - a\int_a^\infty \frac{x^9e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)}\,dx\end{aligned}$$

The first integral is the probability that a Gamma distribution with shape parameter $\alpha = 11$ exceeds a.

For the largest eight samples, we have

x_i	$P(X_i > 2098300)$	$\int_a^\infty (x-a)\frac{x^9e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)}\,dx$
\$ 5,031,900	0.01063134	2.936478e+06
\$ 2,528,000	0.32125681	5.602967e+05
\$ 2,200,600	0.48273624	3.251799e+05
\$ 1,511,800	0.88477086	3.609489e+04
\$ 1,273,400	0.96588111	7.992468e+03
\$ 1,152,400	0.98626386	2.739984e+03
\$ 947,800	0.99861859	2.044913e+02
\$ 789,400	0.99992307	8.766562e+00

From this table, we see that

$$3868995 \leqslant \sum_{i=1}^{n} \int_{a}^{\infty} (x-a) \frac{x^9 e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)} \leqslant 3868995 + 8.766562(n-8)$$

Since

$$\frac{1}{n} \sum_{i=1}^{n} \int_{a}^{\infty} (x-a) \frac{x^9 e^{-\frac{x}{\theta_i}}}{(\theta_i)^{10}\Gamma(10)} \, dx = 82310$$

this becomes

$$\frac{3868995}{n} \leqslant 82310 \leqslant \frac{3868924.8675}{n} + 8.766562$$

$$\frac{3868995}{82310} \leqslant n \leqslant \frac{3868924.8675}{82310} + \frac{8.766562}{82310} n$$

$$47.0051634066 \leqslant n \leqslant 47.0043113534 + 0.000106506645608n$$

$$0.999893493354n \leqslant 47.0043113534$$

$$n \leqslant 47.0093181582$$

Thus we see $n = 47$.