ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 7

Model Solutions

Basic Questions

1. *An insurance company has an insurance policy where the loss amount follows a Pareto distribution with $\alpha = 3.4$ and $\theta = 1000$. Calculate the expected payment per claim if the company introduces a deductible of d.*

   The survival function of the Weibull distribution is $S(x) = \left(\frac{1000}{x+1000}\right)^{3.4}$, so the expected payment per loss with the deductible is

$$\begin{aligned}\int_d^\infty S(x)\,dx &= \int_d^\infty \left(\frac{1000}{x+1000}\right)^{3.4} dx\\ &= \int_{d+1000}^\infty 1000^{3.4}u^{-3.4}\,du\\ &= \left[-1000^{3.4}\frac{u^{-4.4}}{4.4}\right]_{d+1000}^\infty\\ &= \frac{1000^{3.4}}{4.4\times(1000+d)^{4.4}}\end{aligned}$$

   The probability of a claim if the deductible is d is $S(d) = \left(\frac{1000}{d+1000}\right)^{3.4}$, so the expected payment per claim is

$$\frac{\left(\frac{1000^{3.4}}{4.4\times(1000+d)^{4.4}}\right)}{\left(\frac{1000}{d+1000}\right)^{3.4}} = \frac{1}{4.4(d+1000)}$$

2. *The severity of a loss on a worker's compensation insurance policy follows a gamma distribution with $\alpha = 0.3$ and $\theta = 10000$. Calculate the loss eliminatrion ratio of a deductible of \$5,000.*

   Without the deductible, the expected payment per loss is $0.3 \times 10000 =$

$3,000. With the deductible, the expected payment is

$$\begin{aligned}
\int_d^\infty (x-d)\frac{x^{-0.7}e^{-\frac{x}{10000}}}{10000^{0.3}\Gamma(0.3)}\,dx &= \int_d^\infty \frac{x^{0.3}e^{-\frac{x}{10000}}}{10000^{0.3}\Gamma(0.3)}\,dx - d\int_d^\infty \frac{x^{-0.7}e^{-\frac{x}{10000}}}{10000^{0.3}\Gamma(0.3)}\,dx \\
&= 3000\int_d^\infty \frac{x^{0.3}e^{-\frac{x}{10000}}}{10000^{1.3}\Gamma(1.3)}\,dx - d\int_d^\infty \frac{x^{-0.7}e^{-\frac{x}{10000}}}{10000^{0.3}\Gamma(0.3)}\,dx \\
&= 1274.438
\end{aligned}$$

Therefore the loss elimination ratio is

$$1-\frac{1274.438}{3000} = 57.52\%$$

3. *An insurance company has a policy where losses follow an inverse Pareto distribution with $\tau = 1$ and $\theta = 6000$. The companys wants the TVaR at the 95% level for this policy to be $150,000. What policy limit should the company put on the policy to achieve this?*

The distribution function of the inverse Pareto distribition is $F(x) = \left(\frac{x}{x+\theta}\right)^\tau$. The VaR at the 95% level is therefore obtained by solving

$$\begin{aligned}
\frac{x}{x+6000} &= 0.95 \\
\frac{6000}{x+6000} &= 0.05 \\
\frac{x}{6000}+1 &= 20 \\
x &= 114000
\end{aligned}$$

With limit u, the TVaR is

$$\begin{aligned}
\mathrm{TVaR}_{0.95}(X) &= \mathrm{VaR}_{0.95}(X) + 20\int_{\mathrm{VaR}_{0.95}(X)}^{u} S(x)\,dx \\
&= \mathrm{VaR}_{0.95}(X) + 20\int_{\mathrm{VaR}_{0.95}(X)}^{u} \left(1-\frac{x}{x+6000}\right)dx \\
&= \mathrm{VaR}_{0.95}(X) + 20\int_{\mathrm{VaR}_{0.95}(X)}^{u} \left(\frac{6000}{x+6000}\right)dx \\
&= \mathrm{VaR}_{0.95}(X) + 20\int_{6000+\mathrm{VaR}_{0.95}(X)}^{u+6000} 6000v^{-1}\,dv \\
&= \mathrm{VaR}_{0.95}(X) + 20\left[6000\log(v)\right]_{6000+\mathrm{VaR}_{0.95}(X)}^{u+6000} \\
&= 114000 + 20\times 6000\log\left(\frac{u+6000}{120000}\right)
\end{aligned}$$

where we have used the substitution $v = x + 6000$. We therefore need to solve

$$\begin{aligned}
114000 + 120000 \log\left(\frac{u+6000}{120000}\right) &= 150000 \\
\log\left(\frac{u+6000}{120000}\right) &= 0.3 \\
\frac{u+6000}{120000} &= e^{0.3} \\
u &= 120000e^{0.3} - 6000 \\
&= 155983.05691
\end{aligned}$$

4. *Aggregate payments have a compound distribution. The frequency distribution is negative binomial with $r = 2.2$ and $\beta = 3.5$. The severity distribution has mean 2,298 and variance 62,840,000. Use a Pareto approximation to aggregate payments to estimate the probability that aggregate payments are more than 70,000.*

   The frequency distribution has mean $2.2 \times 3.5 = 7.7$ and variance $2.2 \times 3.5 \times 4.5 = 34.65$. Therefore the aggregate loss distribution has mean $7.7 \times 2298 = 17694.6$ and variance $7.7 \times 62840000 + 34.65 \times 2298^2 = 666847858.6$. Setting these equal to the mean and variance of a Pareto distribution with parameters α and θ gives

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 17694.6 \\
\frac{\alpha\theta}{(\alpha-1)^2(\alpha-2)} &= 666847858.6 \\
\frac{\alpha}{\alpha-2} &= \frac{666847858.6}{17694.6^2} = 2.12983157809 \\
1 - \frac{2}{\alpha} &= 0.469520693696 \\
\alpha &= 3.77017534187 \\
\theta &= 17694.6 \times 2.77017534187 = 49017.1446043
\end{aligned}$$

   For these parameters, the probability that payments exceed \$70,000 is

$$\left(\frac{49017.1446043}{49017.1446043 + 70000}\right)^{3.77017534187} = 0.0352773963636$$

Standard Questions

5. *For a certain insurance policy, losses follow an inverse Pareto distribution with $\tau = 4$ and $\theta = 5,000$. The policy limit of \$1,000,000 is applied before the deductible. The deductible is set to achieve a loss elimination ratio of 20%. What deductible achieves this loss elimination ratio?*

   (i) 1246.75

   (ii) 9145.50

   (iii)14547.20

   (iv) 21335.65

   Justify your answer.

   Without the deductible, the expected payment per loss is

$$\begin{aligned}\int_0^u S(x)\,dx &= \int_0^u \left(1-\left(\frac{x}{x+\theta}\right)^4\right)\,dx\\ &= u - \int_\theta^{u+\theta} (v-\theta)^4 v^{-4}\,dv\\ &= u - \int_\theta^{u+\theta} \left(1-4\theta v^{-1}+6\theta^2v^{-2}-4\theta^3v^{-3}+\theta^4v^{-4}\right)\,du\\ &= 4\theta\log\left(\frac{u+\theta}{\theta}\right)-6\theta^2\left(\frac{1}{\theta}-\frac{1}{u+\theta}\right)+4\frac{\theta^3}{2}\left(\frac{1}{\theta^2}-\frac{1}{(u+\theta)^2}\right)-\frac{\theta^4}{3}\left(\frac{1}{\theta^3}-\frac{1}{(u+\theta)^3}\right)\\ &= 84548.4379122\end{aligned}$$

   The reduction in expected payment by introducing a deductible d is

$$\begin{aligned}\int_0^d S(x)\,dx &= 4\theta\log\left(\frac{d+\theta}{\theta}\right)-6\theta^2\left(\frac{1}{\theta}-\frac{1}{d+\theta}\right)+4\frac{\theta^3}{2}\left(\frac{1}{\theta^2}-\frac{1}{(d+\theta)^2}\right)-\frac{\theta^4}{3}\left(\frac{1}{\theta^3}-\frac{1}{(d+\theta)^3}\right)\\ &= 20000\log\left(\frac{d+5000}{5000}\right)-\frac{65000}{3}+\frac{6\theta^2}{\theta+d}-\frac{2\theta^3}{(\theta+d)^2}+\frac{\theta^4}{3(\theta+d)^3}\end{aligned}$$

   We therefore want to set

$$20000\log\left(\frac{d+5000}{5000}\right)-\frac{65000}{3}+\frac{6\theta^2}{\theta+d}-\frac{2\theta^3}{(\theta+d)^2}+\frac{\theta^4}{3(\theta+d)^3} = 0.2\times 84548.4379122 = 16909.6875824$$

   Letting $v = \frac{d}{\theta}$, this becomes

$$4\log(v+1) - \frac{13}{3} + \frac{6}{v+1} - \frac{2}{(v+1)^2} + \frac{1}{3(v+1)^3} = 3.38193751648$$

$$4\log(w) + \frac{6}{w} - \frac{2}{w^2} + \frac{1}{3w^3} = 7.71527084981$$

We try the given values of d to see which one works:

d	w	$4\log(w) + \frac{6}{w} - \frac{2}{w^2} + \frac{1}{3w^3} - 7.71527084981$
(i) 1246.75	1.24935	-3.13267894221
(ii) 9145.50	2.82910	-1.66978069434
(iii)14547.20	3.90944	-0.852227078332
(iv) 21335.65	5.26713	$1.94981207446 \times 10^{-06}$

So (iv) $d = 21335.65$ achieves the desired loss elimination ratio.

6. *An insurance company models loss frequency as negative binomial with $r = 4$ and $\beta = 2.8$, and loss severity as Pareto with $\alpha = 1$, and $\theta = 100$. The insurer wants to set a policy limit u per loss. The insurer buys stop-loss reinsurance for aggregate losses above 1.1 times the expected aggregate losses, the price for which is based on using a Pareto distribution for aggregate losses with parameters fitted using the method of moments. The insurer's loading is 20% for the whole policy, including the ceded part. The stop-loss insurance has a loading of 30%, and the insurer wants to ensure that no more than 25% of its total premiums are paid to the reinsurer. What is the largest value of u they can set to achieve this?*

   (i) $u = \$53,140.43$

   (ii) $u = \$119,243.31$

   (iii) $u = \$160,186.66$

   (iv) $u = \$290,424.04$

   Justify your answer.

   The negative binomial distribution has mean $4 \times 2.8 = 11.2$ and variance $4 \times 2.8 \times 3.8 = 42.56$. If the expected payment per loss is a and the variance is b, then the expected aggregate loss is $11.2a$ and the variance is $11.2b + 42.56a^2$. With a loading of 20%, the aggregate premiums of the insurer are $11.2a \times 1.2 = 13.44a$, so the insurer wants to ensure that the reinsurer's premium is at most $0.25 \times 13.44a = 3.36a$. Since the reinsurer has a loading of 30%, this means the expected payment of the reinsurer must be $\frac{3.36a}{1.3} = 2.58461538462a$.

   The parameters of the Pareto distribution for aggregate losses are set by solving.

$$
\begin{aligned}
\frac{\theta}{\alpha-1} &= 11.2a \\
\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} &= 11.2b + 42.56a^2 \\
\frac{\alpha}{\alpha-2} &= \frac{11.2b+42.56a^2}{(11.2a)^2} \\
&= \frac{1}{11.2}\frac{b}{a^2} + 0.339285714286 \\
1-\frac{2}{\alpha-2} &= \frac{a^2}{0.0892857142857b + 0.339285714286a^2} \\
\frac{2}{\alpha-2} &= \frac{0.0892857142857b - 0.660714285714a^2}{0.0892857142857b + 0.339285714286a^2} \\
\alpha &= 2 + \frac{0.178571428571b + 0.678571428572a^2}{0.0892857142857b - 0.660714285714a^2} \\
&= 2 + \frac{0.178571428571b + 0.678571428572a^2}{0.0892857142857b - 0.660714285714a^2} \\
&= \frac{0.357142857142b - 0.642857142858a^2}{0.0892857142857b - 0.660714285714a^2} \\
\theta &= 11.2\left(\frac{0.357142857142b - 0.642857142858a^2}{0.0892857142857b - 0.660714285714a^2} - 1\right)a \\
&= \frac{3b+0.2a^2}{0.0892857142857b - 0.660714285714a^2}a
\end{aligned}
$$

For a Pareto distribution with parameters α and θ, the expected payment on the stop-loss reinsurance with attachment point $r = 1.1\theta$ is

$$
\begin{aligned}
\int_r^\infty \left(\frac{\theta}{x+\theta}\right)^\alpha dx &= \int_{r+\theta}^\infty \theta^\alpha u^{-\alpha}\,du \\
&= \theta^\alpha\left[\frac{u^{1-\alpha}}{1-\alpha}\right]_{r+\theta}^\infty \\
&= \theta^\alpha\frac{(r+\theta)^{1-\alpha}}{\alpha-1} \\
&= \frac{\theta^\alpha}{(\alpha-1)(r+\theta)^{\alpha-1}} \\
&= \theta\frac{2.1^{1-\alpha}}{(\alpha-1)}
\end{aligned}
$$

Thus, we want to solve

$$\begin{aligned}
\frac{\theta}{\alpha-1}2.1^{1-\alpha} &= 2.58461538462a\\
2.1^{1-\alpha} &= 0.23076923077\\
\alpha &= 5.07983886856\\
\frac{0.357142857142b - 0.642857142858a^2}{0.0892857142857b - 0.660714285714a^2} &= 5.07983886856\\
0.357142857142b - 0.642857142858a^2 &= 5.07983886856(0.0892857142857b - 0.660714285714a^2)\\
0.0964141846694b &= 2.71346496672a^2\\
b &= 28.143835633a^2
\end{aligned}$$

Thus the expected squared payment per loss is $a^2 + b = 29.143835633a^2$.

For the Pareto distribution, the expected payment per loss is

$$\begin{aligned}
a &= \int_0^u \frac{\theta}{(\theta+x)}\,dx\\
&= \int_\theta^{u+\theta} \theta v^{-1}\,dv\\
&= [\theta^\alpha \log(v)]_\theta^{u+\theta}\\
&= \theta \log\left(\frac{u+\theta}{\theta}\right)
\end{aligned}$$

and the expected squared payment per loss is

$$\begin{aligned}
a^2 + b &= \int_0^u 2x\frac{\theta}{(\theta+x)}\,dx\\
&= 2\int_\theta^{u+\theta} (v-\theta)\theta v^{-1}\,dv\\
&= 2\theta\,[v - \theta\log(v)]_\theta^{u+\theta}\\
&= 2\theta\left(u - \theta\log\left(\frac{u+\theta}{\theta}\right)\right)
\end{aligned}$$

We therefore need to choose u such that

$$\begin{aligned}
2\theta\left(u - \theta\log\left(\frac{u+\theta}{\theta}\right)\right) &= 29.143835633\theta^2 \log\left(\frac{u+\theta}{\theta}\right)^2\\
\frac{u}{\theta} &= 2\log\left(1+\frac{u}{\theta}\right) + 29.143835633\log\left(1+\frac{u}{\theta}\right)^2
\end{aligned}$$

Testing the solutions given, we see that (iii) $u = 160186.66$ satisfies this.