ACSC/STAT 3720, Life Contingencies I
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Toby Kenney
Formula Sheet

Notation

For any age, the notation $[x] + s$ indicates current age $x + s$, and select at age x.

- ${}_tp_x$ probability that a life aged x survives for t years.
- ${}_tq_x$ probability that a life aged x dies within t years.
- ${}_{u|t}q_x$ probability that a life aged x survives u years, then dies within the following t years.
- $\mathring{e}_x$ expected future lifetime for a life aged x.
- e_x curtate expected future lifetime for a life aged x.
- $\mathring{e}_{x:\overline{t}|}$ expected future lifetime for a life aged x with upper bound of t.
- i Effective annual interest rate
- v Annual discount factor $(1 + i)^{-1}$
- δForce of interest $\log(1 + i)$
- $i^{(p)}$ Nominal interest rate compounded p times per year
- d Annual discount rate $1 - v$
- $d^{(m)}$ Nominal discount rate compounded m times per year $m(1 - v^{\frac{1}{m}})$
- $\overline{A}_x$ Expected present value of \$1 when a life of present age x dies
- A_x Expected present value of \$1 at the end of the year in which a life of present age x dies
- $A_x^{(m)}$ Expected present value of \$1 at the end of the period $\frac{1}{m}$th of a year in which a life of present age x dies
- ${}^2A_x$ Like A_x, but evaluated at twice the actual force of interest, or effective interest rate $(1 + i)^2 - 1$.
- $A_{x:\overline{t}|}$ Expected present value of \$1 at the end of the year in which a life of present age x dies, or after t years, whichever comes sooner.
- $A^1_{x:\overline{t}|}$ Expected present value of \$1 at the end of the year in which a life of present age x dies provided this happens within t years.
- $u|A_x$ Expected present value of \$1 at the end of the year in which a life of present age x dies provided this happens after at least u years.
- $\ddot{a}_x$ EPV of an annual annuity due with \$1 payments lasting until a life aged x dies. (First payment now)

- a_x EPV of an immediate annual annuity with \$1 payments lasting until a life aged x dies. (First payment in 1 year's time).
- $\ddot{a}_{x:\overline{n}|}$ EPV of an annual annuity due with \$1 payments lasting until a life aged x dies or for a maximum of n payments if the life survives long enough. (First payment now)
- $\ddot{a}_{\overline{n}|}$ EPV of an annual annuity due with \$1 payments lasting for n payments. (First payment now)
- $\ddot{a}_x^m$ EPV of an annuity due with payments $\frac{1}{m}$, m times per year lasting until a life aged x dies. (First payment now)
- $\overline{a}_x$ EPV of an annuity due with continuous payments at a rate of \$1 per year lasting until a life aged x dies.

Formulae

Relations between probabilities

$$
\begin{aligned}
{}_tp_x + {}_t q_x &= 1 \\
{}_u|_tq_x &= {}_u p_x - {}_{u+t} p_x \\
{}_{u+t}p_x &= {}_u p_x {}_tp_{x+u} \\
\mu_x &= -\frac{1}{{}_xp_0}\frac{d}{dx}({}_xp_0) \\
f_x(t) &= {}_t p_x \mu_{x+t} \\
{}_tq_x &= \int_0^t {}_sp_x \mu_{x+s} ds
\end{aligned}
$$

Annuity-Certain

$$
\begin{aligned}
a_{\overline{n}|i} &= \frac{1-(1+i)^{-n}}{i} \\
\ddot{a}_{\overline{n}|i} &= \frac{1-(1+i)^{-n}}{d} \\
s_{\overline{n}|i} &= \frac{(1+i)^n - 1}{i}
\end{aligned}
$$

Formulae for Present Value of a Whole-Life Annuity-due

$$
\begin{aligned}
\ddot{a}_x &= \frac{1-A_x}{d} \\
\ddot{a}_x &= \sum_{k=0}^{\infty} v^k {}_kp_x \\
\ddot{a}_x &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} {}_k|q_x
\end{aligned}
$$

Formulae for Present Value of a Whole-Life Continuous Annuity

$$\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$$
$$\overline{a}_x = \int_{t=0}^{\infty} e^{-\delta t}\,{}_tp_x$$
$$\overline{a}_x = \int_{t=0}^{\infty} \overline{a}_{\overline{t}|\,k}|q_x$$

Relations between Values of Insurance and Annuities

$$\overline{A}_{x:\overline{n}|} = \overline{A}_x + {}_np_x(1+i)^{-n}(1-\overline{A}_{x+n})$$
$$\overline{A}^{1}_{x:\overline{n}|} = \overline{A}_x - {}_np_x(1+i)^{-n}\overline{A}_{x+n} = \overline{A}_{x:\overline{n}|} -_n p_x(1+i)^{-n}$$
$$\overline{a}_{x:\overline{n}|} = \overline{a}_x - {}_np_x(1+i)^{-n}\overline{a}_{x+n} A_{x:\overline{n}|} \qquad = A_x + {}_np_x(1+i)^{-n}(1-A_{x+n})$$
$$A^{1}_{x:\overline{n}|} = A_x - {}_np_x(1+i)^{-n}A_{x+n} = A_{x:\overline{n}|} -_n p_x(1+i)^{-n}$$
$$a_{x:\overline{n}|} = a_x - {}_np_x(1+i)^{-n}a_{x+n} A^{(m)}_{x:\overline{n}|} \qquad = A^{(m)}_x + {}_np_x(1+i)^{-n}(1-A^{(m)}_{x+n})$$
$$A^{(m)}\,{}^{1}_{x:\overline{n}|} = A^{(m)}_x - {}_np_x(1+i)^{-n}A^{(m)}_{x+n} = A^{(m)}_{x:\overline{n}|} -_n p_x(1+i)^{-n}$$
$$a^{(m)}_{x:\overline{n}|} = a^{(m)}_x - {}_np_x(1+i)^{-n}a^{(m)}_{x+n}$$

Policy Values

$${}_tV = (p_{x+t\,t+1}V + q_{x+t}S)(1+i)^{-1} - P$$
$$\frac{d}{dt}\,{}_tV = \delta_t\,{}_tV + P_t - (S_t - {}_tV)\mu_{x+t}$$

where P is the premium payable at time t and S is the death benefit.

Approximations

Uniform Distribution of Deaths (UDD)

Continous case:

$$\overline{A}_x = \frac{i}{\delta}A_x$$

Discrete case:

$$A^m_x = \frac{i}{i^m}A_x$$

Woolhouse's formula

Continuous case:

$$\overline{a}_x = \ddot{a}_x - \frac{1}{2} + \frac{1}{12}(\delta + \mu_x)$$

Discrete case:

$$\ddot{a}^{(m)}_x = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

We often use the approximation $\mu_x = \frac{1}{2}(q_{x-1} + q_x)$.