ACSC/STAT 3720, Life Contingencies I
Winter 2015
Toby Kenney
Homework Sheet 2
Model Solutions

Basic Questions

1. *Using the select lifetable in Table 1, calculate:*

   (a) the probability that an individual aged 43 who was select 1 year ago dies within the next 6 years.

   $l_{[42]+1} = 9940.66$ and $l_{[42]+7} = l_{49} = l_{[46]+3} = 9897.94$, so the probability of surviving is is $\frac{9897.94}{9940.66} = 0.9957025$. The probability of dying is $1 - 0.9957025 = 0.0042975$.

   (b) the probability that an individual aged 49 who was select 7 years ago dies within the next 5 years.

   $l_{[42]+7} = l_{49} = l_{[46]+3} = 9897.94$, and $l_{[42]+12} = l_{54} = l[51] + 3 = 9838.38$, so the probability of surviving 5 years is $\frac{9838.38}{9897.94}$, and the probability of dying within 5 years is $1 - \frac{9838.38}{9897.94} = \frac{59.56}{9897.94} = 0.0060174$.

   (c) the probability that an individual aged 31 who is select survives to age 65.

   We have $l_{[31]} = 9986.40$ and $l_{65} = l_{[62]+3} = 9568.61$, so the probability of surviving is $\frac{9568.61}{9986.40} = 0.9581641$.

   (d) ${}_{2|4}q_{[38]+1}$

   This is the probability that an individual aged 39, who was select at age 38 survives 2 more years, then dies withing the next 4 years. That is, the proability that they die between ages 41 and 45. From the table, we read $l_{[38]+1} = 9961.14$, $l_{[38]+3} = 9953.69$ and $l_{[38]+7} = l_{45} = l_{[42]+3} = 9930.38$. We then have

   $${}_{2|4}q_{[38]+1} = \frac{l_{[38]+3} - l_{[38]+7}}{l_{[38]+1}} = \frac{9953.69 - 9930.38}{9961.14} = \frac{23.31}{9961.14} = 0.00234009$$

2. *An individual's mortality follows the select lifetable in Table 1. You are given that for this lifetable, the expected curtate future lifetime for an individual at age 65 in the ultimate part of the model is 29.04. Calculate the expected curtate future lifetime for*

   (a) a select individual aged 60

   The curtate expected future lifetime is given by $\mathring{e}_x =_1 p_x +_2 p_x + cdots$, which in this case is given by $\mathring{e}_x =_1 p_x +_2 p_x +_3 p_x +_4 p_x +_5 p_x +_5 p_x \mathring{e}_{x+5}$. Substituting $x = 60$ gives

   $$\begin{aligned}\mathring{e}_x &= \frac{l_{[60]+1} + l_{[60]+2} + l_{[60]+3} + l_{[60]+4} + l_{[60]+5} + l_{[60]+5}\mathring{e}_{65}}{l_{[60]}} \\ &= \frac{9686.43 + 9665.17 + 9638.51 + 9605.07 + 9568.61 + 9568.61 \times 29.04}{9703.36} \\ &= 33.60\end{aligned}$$

Table 1: Select lifetable to be used for questions on this assignment

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	9998.75	9997.65	9996.30	9994.66	74	8987.73	8932.10	8862.49	8775.52
26	9997.00	9995.83	9994.40	9992.66	75	8897.04	8836.71	8761.27	8667.10
27	9995.14	9993.90	9992.38	9990.52	76	8798.69	8733.34	8651.66	8549.78
28	9993.16	9991.84	9990.22	9988.24	77	8692.13	8621.41	8533.09	8423.00
29	9991.05	9989.65	9987.92	9985.80	78	8576.81	8500.36	8404.95	8286.16
30	9988.81	9987.30	9985.46	9983.18	79	8452.13	8369.60	8266.68	8138.66
31	9986.40	9984.80	9982.82	9980.38	80	8317.52	8228.53	8117.67	7979.93
32	9983.83	9982.11	9979.99	9977.37	81	8172.36	8076.57	7957.35	7809.41
33	9981.07	9979.23	9976.95	9974.13	82	8016.08	7913.13	7785.15	7626.56
34	9978.11	9976.13	9973.68	9970.64	83	7848.11	7737.67	7600.54	7430.89
35	9974.93	9972.79	9970.16	9966.88	84	7667.89	7549.66	7403.05	7221.99
36	9971.50	9969.20	9966.36	9962.82	85	7474.92	7348.64	7192.27	6999.51
37	9967.80	9965.33	9962.25	9958.44	86	7268.77	7134.21	6967.86	6763.22
38	9963.81	9961.14	9957.82	9953.69	87	7049.07	6906.07	6729.62	6513.04
39	9959.50	9956.61	9953.02	9948.55	88	6815.55	6664.05	6477.46	6249.02
40	9954.84	9951.71	9947.82	9942.98	89	6568.09	6408.10	6211.48	5971.42
41	9949.79	9946.41	9942.19	9936.94	90	6306.70	6138.35	5931.96	5680.73
42	9944.32	9940.66	9936.08	9930.38	91	6031.59	5855.15	5639.41	5377.67
43	9938.39	9934.41	9929.45	9923.26	92	5743.19	5559.08	5334.61	5063.27
44	9931.96	9927.64	9922.25	9915.52	93	5442.15	5250.97	5018.61	4738.86
45	9924.97	9920.28	9914.42	9907.10	94	5129.44	4931.97	4692.79	4406.12
46	9917.37	9912.28	9905.91	9897.94	95	4806.33	4603.54	4358.89	4067.08
47	9909.11	9903.58	9896.65	9887.98	96	4474.39	4267.51	4018.96	3724.10
48	9900.13	9894.11	9886.57	9877.13	97	4135.60	3926.04	3675.44	3379.91
49	9890.36	9883.80	9875.59	9865.30	98	3792.25	3581.66	3331.11	3037.57
50	9879.71	9872.57	9863.63	9852.42	99	3447.02	3237.23	2989.05	2700.39
51	9868.12	9860.34	9850.59	9838.38	100	3102.90	2895.94	2652.63	2371.88
52	9855.48	9847.01	9836.39	9823.08	101	2763.19	2561.21	2325.37	2055.64
53	9841.72	9832.48	9820.90	9806.39	102	2431.39	2236.61	2010.90	1755.27
54	9826.71	9816.64	9804.02	9788.18	103	2111.15	1925.80	1712.81	1474.18
55	9810.34	9799.37	9785.60	9768.33	104	1806.12	1632.34	1434.48	1215.44
56	9792.49	9780.52	9765.51	9746.67	105	1519.82	1359.55	1178.94	981.65
57	9773.03	9759.97	9743.60	9723.05	106	1255.46	1110.36	948.70	774.71
58	9751.79	9737.56	9719.69	9697.28	107	1015.81	887.14	745.58	595.71
59	9728.63	9713.10	9693.62	9669.17	108	802.96	691.49	570.56	444.87
60	9703.36	9686.43	9665.17	9638.51	109	618.23	524.17	423.71	321.41
61	9675.80	9657.33	9634.15	9605.07	110	462.04	385.00	304.13	223.65
62	9645.73	9625.59	9600.31	9568.61	111	333.80	272.80	210.00	149.10
63	9612.94	9590.98	9563.42	9528.85	112	231.99	185.53	138.71	94.62
64	9577.18	9553.24	9523.19	9485.52	113	154.19	120.34	87.07	56.74
65	9538.19	9512.09	9479.35	9438.30	114	97.30	73.90	51.50	31.84
66	9495.69	9467.25	9431.58	9386.86	115	57.78	42.55	28.41	16.52
67	9449.37	9418.39	9379.54	9330.85	116	31.92	22.69	14.43	7.81
68	9398.90	9365.17	9322.87	9269.88	117	16.15	11.04	6.63	3.30
69	9343.95	9307.23	9261.20	9203.55	118	7.34	4.79	2.69	1.21
70	9284.12	9244.18	9194.11	9131.43	119	2.90	1.79	0.93	0.37
71	9219.03	9175.59	9121.17	9053.07	120	0.95	0.55	0.26	0.09
72	9148.24	9101.03	9041.91	8967.97	121	0.23	0.13	0.05	0.01
73	9071.30	9020.03	8955.85	8875.63	122	0.03	0.02	0.01	0.00

(b) an individual aged 60 who was select 1 year ago.

This is similar to the previous part, except that the individual was select at age 59, so we get

$$\begin{aligned}\mathring{e}_x &= \frac{l_{[59]+2} + l_{[59]+3} + l_{[59]+4} + l_{[59]+5} + l_{[59]+6} + l_{[59]+6}\mathring{e}_{65}}{l_{[59]+1}} \\ &= \frac{9693.62 + 9669.17 + 9638.51 + 9605.07 + 9568.61 + 9568.61 \times 29.04}{9713.10} \\ &= 33.57\end{aligned}$$

3. *Using the ultimate mortality model*

$$\mu_x = 0.000077 + 0.0000078 \times 1.099^x$$

and select mortality

$$\mu_{[x]+s} = 0.85^{3-s}\mu_{x+s}$$

for the selection period of 3 years, calculate a select life table between ages 41 and 45 using radix 10,000. [You may use the approximation $q_{[x]+t} = \mu_{[x+0.5]+t}$.]

We start by calculating

$$\begin{aligned} q_{41} &= 0.0004691921 & {}_1p_{40} &= 0.9995308 \\ q_{42} &= 0.0005080191 & {}_2p_{40} &= 0.9990230 \\ q_{43} &= 0.0005506900 & {}_3p_{40} &= 0.9984729 \\ q_{44} &= 0.0005975853 & {}_4p_{40} &= 0.9978762 \\ q_{45} &= 0.0006491233 & {}_5p_{40} &= 0.9972285 \\ q_{46} &= 0.0007057635 & {}_6p_{40} &= 0.9965247 \\ q_{47} &= 0.0007680111 & {}_7p_{40} &= 0.9957593 \end{aligned}$$

This gives us the entries in the final column of the lifetable. We then calculate the earlier entries working backwards from these. For example:

$$\begin{aligned} q_{[41]+2} &= 0.85q_{43} = 0.0004680865 & p_{[41]+2} &= 0.9995319 & l_{[41]+2} &= \frac{9984.74}{p_{[41]+2}} = 9989.40 \\ q_{[41]+1} &= 0.85^2q_{42} = 0.0003670438 & p_{[41]+1} &= 0.9996330 & l_{[41]+1} &= \frac{9989.406}{p_{[41]+1}} = 9993.07 \\ q_{[41]} &= 0.85^3q_{41} = 0.0002881426 & p_{[41]} &= 0.9997119 & l_{[41]} &= \frac{9993.074}{p_{[41]}} = 9995.95 \end{aligned}$$

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
41	9995.95	9993.07	9989.40	9984.73
42	9990.92	9987.81	9983.83	9978.76
43	9985.48	9982.10	9977.79	9972.28
44	9979.57	9975.91	9971.23	9965.25
45	9973.16	9969.18	9964.10	9957.59

4. *The lifetable in Table 1 applied 15 years ago. The following is an excerpt from the ultimate part of an updated lifetable:*

x	l_x	d_x
46	*10000.00*	*4.61*
47	*9995.39*	*5.03*
48	*9990.36*	*5.49*
49	*9984.87*	*6.00*
50	*9978.87*	*6.55*

Calculate the reduction factor used for each age between 46 and 50.

The original lifetable had

$$p_{46} = \frac{9915.52}{9923.26} = 0.9992200 \qquad q_{46} = 0.0007799856$$
$$p_{47} = \frac{9907.10}{9915.52} = 0.9991508 \qquad q_{47} = 0.0008491738$$
$$p_{48} = \frac{9897.94}{9907.10} = 0.9990754 \qquad q_{48} = 0.0009245894$$
$$p_{49} = \frac{9887.98}{9897.94} = 0.9989937 \qquad q_{49} = 0.0010062700$$
$$p_{50} = \frac{9877.13}{9887.98} = 0.9989027 \qquad q_{50} = 0.0010972919$$

The new lifetable has

$$q_{46} = \frac{4.61}{10000.00} = 0.0004610000$$
$$q_{47} = \frac{5.03}{9995.39} = 0.0005032320$$
$$q_{48} = \frac{5.49}{9990.36} = 0.0005495297$$
$$q_{49} = \frac{6.00}{9984.87} = 0.0006009092$$
$$q_{50} = \frac{6.55}{9978.87} = 0.0006563869$$

The reduction factors are therefore:

$$\left(\frac{0.0004610000}{0.0007799856}\right)^{\frac{1}{5}} = 0.9001655$$
$$\left(\frac{0.0005032320}{0.0008491738}\right)^{\frac{1}{5}} = 0.9006464$$
$$\left(\frac{0.0005495297}{0.0009245894}\right)^{\frac{1}{5}} = 0.9011736$$
$$\left(\frac{0.0006009092}{0.0010062700}\right)^{\frac{1}{5}} = 0.9020256$$
$$\left(\frac{0.0006563869}{0.0010972919}\right)^{\frac{1}{5}} = 0.9023345$$

Standard Questions

5. *An insurance company determines that individuals aged 34 who do not pass the underwriting process used for policies following Table 1 have probability 0.066 of dying within the next three years. What is the probability that a select individual aged 31 would pass the underwriting process in three years' time (at age 34). [Either an individual passes the underwriting process, in which case their mortality is that of a select life at their age, or they do not pass the underwriting process.]*

   The probability that the individual survives for 6 years from age 31 is

$$_{6}p_{[31]} = P(\text{alive and select at } 34)\,{}_{3}p_{[34]} + P(\text{alive but not select at } 34) \times 0.934$$

   We also have

$$_{3}p_{[31]} = P(\text{alive and select at } 34) + P(\text{alive but not select at } 34)$$

   Letting s be the probability that the individual is alive and passes the underwriting process at age 34, and t be the probability that the individual is alive but fails the underwriting process. Our equations are:

$$\begin{aligned}
\frac{9970.64}{9978.11}s + 0.934t &= \frac{9970.64}{9986.40} \\
s + t &= \frac{9980.38}{9986.40} \\
\left(1 - 0.934 \times \frac{9978.11}{9970.64}\right) t &= \frac{9980.38}{9986.40} - \frac{9978.11}{9986.40} \\
t &= 0.003480984 \\
s &= \frac{9980.38}{9986.40} - 0.003480984 = 0.9959162
\end{aligned}$$

6. *An insurance company has used Makeham's formula with a constant factor to discount for selected lives — that is $\mu_{[x]+s} = D^{3-s}\mu_{x+s}$ to construct a lifetable for female smokers. The lifetable is given below.*

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
50	*9987.68*	*9965.07*	*9937.72*	*9904.65*
51	*9958.00*	*9932.58*	*9901.84*	*9864.68*
52	*9924.63*	*9896.06*	*9861.53*	*9819.78*
53	*9887.12*	*9855.03*	*9816.24*	*9769.36*
54	*9844.98*	*9808.94*	*9765.38*	*9712.77*

Find the parameters used in the model to produce the table. [It was constructed using the approximation $q_{[x]+s} = \mu_{[x+0.5]+s}$.]

From the table, we read $q_{53} = 1 - \frac{9864.68}{9904.65}$ and $q_{[51]+2} = 1 - \frac{9864.68}{9901.84}$, so we get $D = \frac{q_{[51]+2}}{q_{53}} = 0.9300$. We also read

$$\begin{aligned}
A + BC^{53.5} &= \mu_{53.5} = q_{53} = 1 - \frac{9864.68}{9904.65} = 0.004035478 \\
A + BC^{54.5} &= \mu_{54.5} = q_{54} = 1 - \frac{9819.78}{9864.68} = 0.004551592 \\
A + BC^{55.5} &= \mu_{55.5} = q_{55} = 1 - \frac{9769.36}{9819.78} = 0.005134535 \\
A + BC^{56.5} &= \mu_{56.5} = q_{56} = 1 - \frac{9712.77}{9769.36} = 0.005792601
\end{aligned}$$

We therefore get $BC^{53.5}(C-1) = 0.004551592 - 0.004035478$ and $BC^{54.5}(C-1) = 0.005134535 - 0.004551592$, so

$$C = \frac{BC^{53.5}(C-1)}{BC^{54.5}(C-1)} = 1.129484$$

This also gives $A(C-1) = 0.004035478C - 0.004551592 = 0.000006415833$, so $A = 0.00004954924$. Finally, we get $B = \frac{0.004035478 - 0.00004954924}{1.129484^{53.5}} = 0.000005908063$.

[The actual values used are $A = 0.000033$, $B = 0.00000607$, $C = 1.129$ and $D = 0.93$. Depending on how you calculate these values from the table, you could get different values. The values for C and D should be close to the true values, since rounding errors are small. Values for A and B are more influenced by errors in the value of C. Using the correct value of C in my calculations would give $A = 0.00003459428$ and $B = 0.000006067783$.]