ACSC/STAT 4703, Actuarial Models II
Fall 2015
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Homework Sheet 2
Model Solutions

Basic Questions

1. *An insurance company has the following portfolio of home insurance policies:*

Type of driver	*Number*	*Probability claim*	*mean of claim*	*standard deviation*
Small home	*800*	*0.12*	*\$30,000*	*\$15,000*
Medium home	*2100*	*0.07*	*\$55,000*	*\$42,000*
Large home	*500*	*0.06*	*\$100,000*	*\$83,000*

Calculate the cost of reinsuring losses above \$20,000,000, if the loading on the reinsurance premium is one standard deviation above the expected claim payment on the reinsurance policy:

(a) using a gamma approximation for the aggregate losses on this portfolio.

The expected value of the aggregate loss is

$$800\times0.12\times30000+2100\times0.07\times55000+500\times0.06\times100000 = 2880000+8085000+3000000 = 13965000$$

The variance is

$$\begin{aligned}&800 \times 0.12 \times 0.88 \times 30000^2 + 2100 \times 0.07 \times 0.93 \times 55000^2 + 500 \times 0.06 \times 0.94 \times 100000^2\\&\quad + 800 \times 0.12 \times 15000^2 + 2100 \times 0.07 \times 42000^2 + 500 \times 0.06 \times 83000^2\\&= 1259157750000\end{aligned}$$

The parameters of the Gamma distribution that match this are

$$\theta = \frac{1259157750000}{13965000} = 90165.25$$
$$\alpha = \frac{13965000}{90165.25} = 154.8823$$

The expected payment on the stop-loss insurance is therefore

$$\int_{20000000}^{\infty} \frac{x^{154.8823} - 20000000x^{153.8823}}{90165.25^{154.8823}\Gamma(154.8823)} e^{-\frac{x}{90165.25}}\,dx$$
$$= 154.8823 \times \int_{20000000}^{\infty} \frac{x^{154.8823}}{90165.25^{155.8823}\Gamma(155.8823)} e^{-\frac{x}{90165.25}}\,dx$$
$$- 20000000 \int_{20000000}^{\infty} \frac{x^{153.8823}}{90165.25^{154.8823}\Gamma(154.8823)} e^{-\frac{x}{140295.8}}\,dx$$
$$= 0.2431946$$

The expected square of the payment is

$$\int_{20000000}^{\infty} \frac{x^{155.8823} - 40000000x^{154.8823} + 20000000^2 x^{153.8823}}{140295.8^{154.8823}\Gamma(154.8823)} e^{-\frac{x}{140295.8}}\,dx$$
$$= 154.8823 \times 155.8823 \times 140295.8^2 \int_{20000000}^{\infty} \frac{x^{155.8823}}{140295.8^{51.64511}\Gamma(51.64511)} e^{-\frac{x}{140295.8}}\,dx$$
$$- 154.8823 \times 140295.8 \times 40000000 \int_{20000000}^{\infty} \frac{x^{154.8823}}{140295.8^{155.8823}\Gamma(155.8823)} e^{-\frac{x}{140295.8}}\,dx$$
$$+ 20000000^2 \int_{20000000}^{\infty} \frac{x^{153.8823}}{140295.8^{154.8823}\Gamma(154.8823)} e^{-\frac{x}{140295.8}}\,dx$$
$$= 131897.6$$

So the variance is

$$131897.6 - 0.2431946^2 = 131897.5$$

The standard deviation is 363.177, so the premium is $15.99077+363.177 =$ \$379.17.

(b) using a compound Poisson approximation, with Poisson mean set to match the probability of no loss, and modelling the loss distribution for each type as a distribution taking only two values with probability 0.5 — that is, for example, for small homes, the claim amount is either \$15,000 or \$45,000 with probability 0.5 for each, so that the mean and standard deviation are as in the table. [You may use the fact that for a standard normal, the excess-of-loss random variable $(Z-a)_+$ *has mean* $\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - a(1-\Phi(a))$*, and variance* $(1-\Phi(a))\left(a^2\Phi(a) + 2a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} + 1\right) + a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-a^2}}{2\pi}$*.]*

If the probability of no loss is $1-p$, then the Poisson parameter is the solution to $e^{-\lambda} = 1-p$, so $\lambda = -\log(1-p)$. The total number of claims is therefore approximated as a Poisson distribution with

$\lambda = -800\log(0.88) - 2100\log(0.93) - 500\log(0.94) = 285.6029$. The secondary distribution has the following probabilities:

Loss	Probability
15,000	0.1790365
45,000	0.1790365
13,000	0.2668013
97,000	0.2668013
17,000	0.05416209
183,000	0.05416209

This secondary distribution has mean 50922.75 and variance 2194614913, so using a normal approximation for the aggregate of N total claims, the total of N losses has mean $50922.75N$ and variance $2194614913N$.

Recall that the mean excess-of-loss for a standard normal, with deductible a is

$$\begin{aligned}
\int_a^\infty \frac{(x-a)}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx &= \frac{1}{\sqrt{2\pi}}\left(\int_a^\infty xe^{-\frac{x^2}{2}}\,dx - a\int_a^\infty e^{-\frac{x^2}{2}}\,dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_a^\infty - a(1-\Phi(a))\\
&= \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - a(1-\Phi(a))
\end{aligned}$$

The expected square excess-of-loss for a standard normal, with deductible a is

$$\begin{aligned}
\int_a^\infty \frac{(x-a)^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx &= \frac{1}{\sqrt{2\pi}}\left(\int_a^\infty x^2e^{-\frac{x^2}{2}}\,dx - 2a\int_a^\infty xe^{-\frac{x^2}{2}}\,dx + a^2\int_a^\infty xe^{-\frac{x^2}{2}}\,dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\left[-xe^{-x^2}\right]_a^\infty + \int_a^\infty e^{-x^2}\,dx\right) - \frac{2a}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_a^\infty + a^2(1-\Phi(a))\\
&= (a^2+1)(1-\Phi(a)) - a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}
\end{aligned}$$

The variance of this excess-of-loss random variable is therefore

$$(a^2+1)(1-\Phi(a)) - a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - \left(\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - a(1-\Phi(a))\right)^2$$

$$= (a^2+1)(1-\Phi(a)) - a^2(1-\Phi(a))^2 + 2a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}(1-\Phi(a)) + a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - \left(\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\right)^2$$

$$= (1-\Phi(a))\left(a^2\Phi(a) + 2a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} + 1\right) + a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-a^2}}{2\pi}$$

For our case, the mean excess-of-loss conditional on N claims is

$$L_N = \sqrt{2194614913N}\left(\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - a(1-\Phi(a))\right)$$

where $a = \frac{20000000-50922.75N}{\sqrt{2194614913N}}$.
Therefore the total expected excess of loss is

$$e^{-285.6029}\sum_{n=0}^{\infty}\frac{285.6029^n}{n!}L_n = \$1.397723$$

and the expected square of the conditional expectation of excess of loss is

$$e^{-285.6029}\sum_{n=0}^{\infty}\frac{285.6029^n}{n!}{L_n}^2 = 11358.4$$

The conditional variance of excess-of-loss is

$$V_N = 2194614913N\left((1-\Phi(a))\left(a^2\Phi(a) + 2a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} + 1\right) + a\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-a^2}}{2\pi}\right)$$

The expected conditional variance of excess-of-loss is therefore

$$e^{-285.6029}\sum_{n=0}^{\infty}\frac{285.6029^n}{n!}V_n = \$66,531,806$$

So the total variance of aggregate excess of loss is

$$11358.4 + 66531806 = 66543164$$

which makes the standard deviation

$$\sqrt{66543164} = \$8,157.40$$

The premium is therefore $1.40 + 8,157.40 = \$8,158.80$.

2. *Calculate the values Emprirical distribution function and the cumulative hazard rate function (using a Nelson-Åalen) at $x = 2.85$ for the following dataset*

```
2.2 3.4 1.6 2.8 1.3 2.2 3.3 2.8 3.1 2.0 1.3 3.3 2.5
1.9 2.0 1.8 2.4 2.9 3.1
```

We sort the sample into order:

```
1.3 1.3 1.6 1.8 1.9 2.0 2.0 2.2 2.2 2.4 2.5 2.8 2.8
2.9 3.1 3.1 3.3 3.3 3.4
```

The empirical distribution function is just the proportion of data points below x. In particular, we have $F(2.85) = \frac{13}{19} = 0.6842105$.

The cumulative hazard rate is estimated as the sum over all observed values of X, of the ratio of number of times the value is observed, divided by number of times a larger value is observed. In this case:

$$\hat{H}(2.85) = \frac{2}{19} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{2}{14} + \frac{2}{12} + \frac{1}{10} + \frac{1}{9} + \frac{2}{8} = 1.063888$$

3. *For the sample from Question 2, calculate a Nelson-Åalen estimate for the probability that a random sample is more than 2.7.*

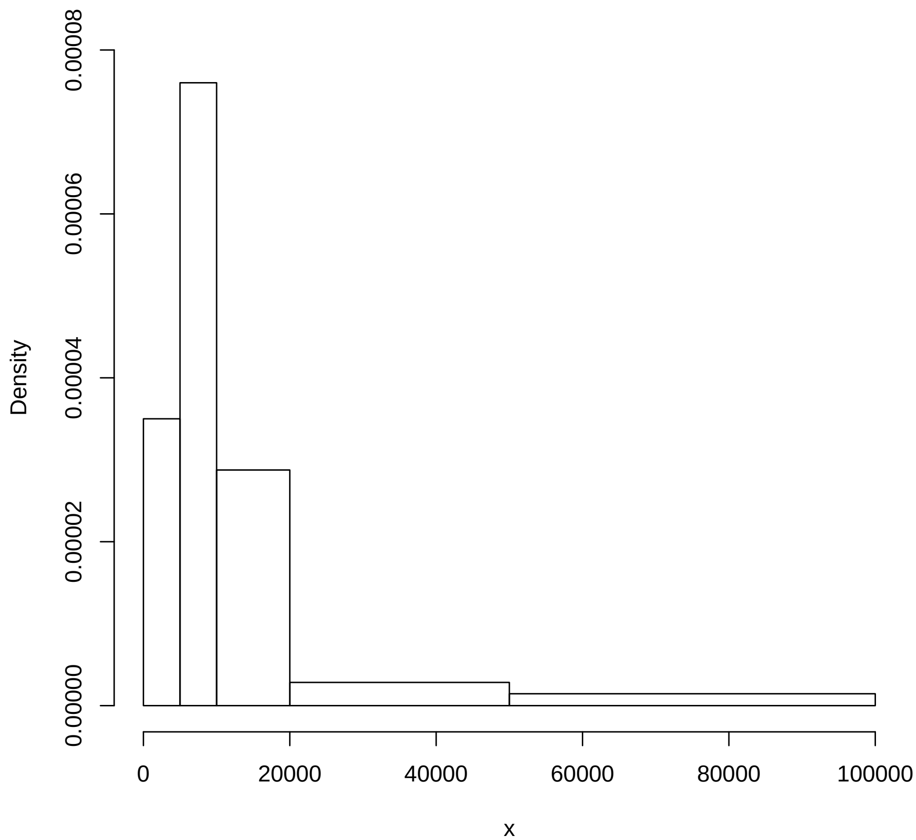
We first estimate the cumulative hazard rate funtion at 2.7, which is

$$\hat{H}(2.7) = \frac{2}{19} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{2}{14} + \frac{2}{12} + \frac{1}{10} + \frac{1}{9} = 0.813888$$

The survival function estimate is $e^{-\hat{H}(2.7)} = e^{-0.813888} = 0.4431318$.

4. *Draw a histogram of the following distribution:*

Claim Amount	*Number of Claims*
Less than \$5,000	*70*
\$5,000–\$10,000	*152*
\$10,000–\$20,000	*115*
\$20,000–\$50,000	*34*
\$50,000–\$100,000	*29*



Standard Questions

5. *In a group life insurance policy, the insurer divides lives into three categories: "preferred", "standard" and "impaired". The probabilities of dying for individuals in the three categories are* 0.002, 0.005 *and* 0.015 *respectively. The group currently includes 18 preferred lives, 497 standard lives and 46 impaired lives. All lives have the same death benefit. The premium for each life is set so that aggregate premiums are one standard deviation above the mean for aggregate claims. An additional 20 impaired lives are added to the group. By what percentage should the premium per life increase?*

   The number of deaths before the new lives join the group can be approximated by a Poisson distribution with $\lambda = 0.002 \times 18 + 0.005 \times 497 +$

$0.015 \times 46 = 0.036 + 2.485 + 0.69 = 3.221$. If the benefit is D, then the premium per life is $\frac{3.221+\sqrt{3.221}}{561} D = 0.008940667D$.

After the new lives join the group, the number of deaths follows a Poisson distribution with $\lambda = 0.002 \times 18 + 0.005 \times 497 + 0.015 \times 66 = 0.036 + 2.485 + 0.99 = 3.521$. If the benefit is D, then the premium per life is $\frac{3.521+\sqrt{3.521}}{581} D = 0.009289902D$.

The percentage increase is therefore $\frac{0.009289902}{0.008940667} - 1 = 3.906\%$.

6. *An insurance company collects the following data on insurance claims:*

Claim Amount	*Number of Policies*
Less than \$5,000	*230*
\$5,000–\$10,000	*152*
\$10,000–\$50,000	*112*
\$50,000–\$200,000	*44*
\$200,000–\$1,000,000	*64*
More than \$1,000,000	*16*

The policy currently has no deductible and a policy limit of \$1,000,000. The company wants to determine how much would be saved by introducing a deductible of \$2,000 and a policy limit of \$500,000. Using the ogive to estimate the empirical distribution, how much would the expected claim amount per loss be reduced by the new deductible and policy limit?

With the new policy limit, the 16 claims for more than \$1,000,000 will also be reduced by \$500,000. Under the ogive, the number of claims between \$500,000 and \$1,000,000 is $\frac{500000}{800000} \times 64 = 40$. The average amount by which one of these claims is reduced is \$250,000, so the total reduction in aggregate claims is \$18,000,000. This is over 618 claims, so the reduction in expected loss is $\frac{18000000}{618} = \$29,126.21$. Using the ogive, there are 92 claims between \$0 and \$2,000. These will be reduced by \$1,000 on average, while the 526 claims above \$2,000 will all be reduced by \$2,000, for a total reduction of $92000 + 1052000 = \$1,144,000$. This is a reduction of $\frac{1144000}{618} = \$1,851.13$ per loss. The total reduction in expected claim per loss is therefore $29126.21 + 1851.13 = \$30,977.35$.

7. *An insurance company calculates the expected payment per claim as \$11,342 based on a sample of 3,000 claims with a policy limit of \$1,000,000. The company finds that these claims included 19 claims which exceeded the policy limit. The loss amounts for these claims were:*

1,263,000	*4,712,000*	*1,630,000*	*1,103,000*	*2,241,000*	*1,420,000*
1,337,000	*1,831,000*	*1,903,000*	*1,557,000*	*1,670,000*	*1,098,000*
1,226,000	*2,947,000*	*1,346,000*	*1,660,000*	*1,403,000*	*1,185,000*
1,631,000					

Calculate the expected payment per claim if the policy limit is increased to \$2,000,000.

The total of all 3000 claims after policy limits were applied was $11342 \times 3000 = \$34,026,000$. Increasing the policy limit to $2,000,000 would increase the aggregate claims by (in thousands)

$$263 + 1000 + 630 + 103 + 1000 + 420 + 337 + 831 + 903 + 557 + 670 + 98 + 226 + 1000 + 346 + 660$$
$$+403 + 185 + 631 = 10263$$

The new aggregate claims are therefore $34026000 + 10263000 = \$44,289,000$. Divided among 3000 claims, the average payment per claim is $\frac{44289000}{3000} = \$14,763$.