ACSC/STAT 4703, Actuarial Models II
Fall 2015
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Homework Sheet 7
Model Solutions

Basic Questions

1. *An insurance company sets the book pure premium for its car insurance premium at \$836. The expected process variance is 342,017 and the variance of hypothetical means is 86,202. If an individual has no claims over the last 6 years, calculate the credibility premium for this individual's next year's insurance using the Bühlmann model.*

   The Bühlmann credibility is $Z = \frac{6}{6+\frac{342017}{86202}} = 0.601949$, so next year's premium is $0.601949 \times 0 + 836 \times (1 - 0.601949) = \$332.77$.

2. *An insurance company has the following data on a group life insurance policy:*

Year	*1*	*2*	*3*	*4*	*5*	*6*
No. insured	*1,204*	*1,320*	*972*	*1,504*	*1,670*	*1,583*
No. of deaths	*8*	*3*	*7*	*9*	*11*	*10*

   The average mortality rate for the population is 1 death per 100 policies per year. The variance of hypothetical means of deaths per policy is 0.0014. Using a Bühlmann-Straub model, calculate the credibility premium for this group insurance for Year 7.

   The variance of hypothetical means is 0.0014. The number of deaths per policy is a Bernoulli random variable where p has mean 0.01 and variance 0.0014, so the variance conditional on p is $p(1-p)$, so the expected process variance is the expected value of $p-p^2$. The expected value of p is 0.01, and the expected value of p^2 is $0.0014+0.01^2 = 0.0015$. The expected process variance is therefore $0.01 - 0.0015 = 0.0085$. The credibility is therefore $\frac{8253}{8253+\frac{0.0085}{0.0014}} = 0.9992649$.

   The expected number of deaths per year is therefore $0.9992649 \times \frac{48}{8253} + 0.0007351 \times 0.01 = 0.005819143$.

3. *An insurance company has the following previous data on aggregate claims:*

Policyholder	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*	*Year 6*	*Mean*	*Variance*
1	*0*	*445*	*0*	*0*	*877*	*1,198*	*420*	*268791.6*
2	*916*	*1,533*	*777*	*0*	*0*	*1,487*	*785.5*	*460396.3*
3	*709*	*0*	*0*	*1,275*	*924*	*0*	*484.6667*	*314534.3*
4	*1,910*	*2,004*	*3,723*	*714*	*1,410*	*422*	*1697.167*	*1383715.4*
5	*0*	*927*	*0*	*0*	*0*	*0*	*153.5*	*143221.5*

   Calculate the Bühlmann credibility premium for each policyholder in Year 7.

   The expected process variance is $\frac{268791.6+460396.3+314534.3+1383715.4+143221.5}{5} = 514131.8$. The mean claim amount is $\frac{420+785.5+484.6667+1697.167+153.5}{5} = 708.1667$, so the variance of observed means is

   $$\frac{(420-708.1667)^2+(785.5-708.1667)^2+(484.6667-708.1667)^2+(1697.167-708.1667)^2+(153.5-708.1667)^2}{4} = 356187.2$$

Since the average conditional variance of observed means is the EPV divided by 6, the VHM is $356187.2 - \frac{514131.8}{6} = 270498.6$

The Bühlmann credibility is therefore $Z = \frac{6}{6+\frac{514131.8}{270498.6}} = 0.7594281$. The credibility premiums are therefore

$$0.7594281 \times 420 + 0.2405719 \times 708.16667 = \$489.32$$
$$0.7594281 \times 785.5 + 0.2405719 \times 708.16667 = \$766.90$$
$$0.7594281 \times 484.6667 + 0.2405719 \times 708.16667 = \$538.43$$
$$0.7594281 \times 1697.167 + 0.2405719 \times 708.16667 = \$1,459.24$$
$$0.7594281 \times 153.5 + 0.2405719 \times 708.16667 = \$286.94$$

4. *Over a three-year period, an insurance company observes the following numbers of claims:*

No. of claims	*Frequency*
0	*3,935*
1	*4,108*
2	*1,420*
3	*637*
4	*211*
5	*94*
6	*40*
7	*15*
8	*4*
9	*0*
10	*1*

Some customers only had a two-year claim history. Claim frequencies for these customers were:

No. of claims	*Frequency*
0	*670*
1	*482*
2	*289*
3	*104*
4	*38*
5	*9*
6	*2*
7	*1*

Assuming the number of claims made by an individual in a year follows a Poisson distribution, calculate the credibility estimate for the expected claim frequency in the following year, of an individual who has made a total of 4 claims in the past 2 years.

The total number of claims observed is

$(4108 + 482) \times 1 + (1420 + 289) \times 2 + (637 + 104) \times 3 + (211 + 38) \times 4 + (94 + 9) \times 5 + (40 + 2) \times 6 + (15 + 1) \times 7 + 4 \times 8 + 1 \times 10 = 12148$

The number of years observed is

$(3935+4108+1420+637+211+94+40+15+4+1)\times 3+(670+482+289+104+38+9+2+1)\times 2 = 34585$

The average number of claims per year is therefore $\frac{12148}{34585} = 0.3512505$.

Since the variance of a Poisson distribution is equal to the mean, this means that the expected process variance is also 0.3512505.

We calculate the total variance of average number of claims per year per person as:

$\frac{1}{12059}\left(3935(0-0.3512505)^2+4108\left(\frac{1}{3}-0.3512505\right)^2+1420\left(\frac{2}{3}-0.3512505\right)^2+637(1-0.3512505)^2+211\left(\frac{4}{3}-0.3512505\right)^2\right.$
0.1643035

[This is the variance per individual, not per policy-year. Per policy-year the variance is 0.1568069.]

The expected process variance of the mean for individuals with a 3-year history is $\frac{0.3512505}{3} = 0.1170835$. For the individuals with two-year histories, the expected process variance for the mean is $\frac{0.3512505}{2} = 0.1756253$. The overall variance due to Poisson sampling is therefore $10465\times 0.1170835+1595\times 0.175625312060 = 0.1248257$. This means the estimated VHM is $0.1643035-0.1248257 = 0.0394778$.

The credibility of an individual with two years past history is therefore $Z = \frac{2}{2+\frac{0.3512505}{0.0394778}} = 0.1835297$. The expected claim frequency is therefore $2\times 0.1835297+0.3512505\times 0.8164703 = 0.653845$.

Using per policy-year, the Poisson variance is $0.3512505\times 12060/34585 = 0.1224832$, so the VHM is $0.1568069-0.1224832 = 0.0343237$. This means the credibility of an individual with two years past history is therefore $Z = \frac{2}{2+\frac{0.3512505}{0.0343237}} = 0.1634859$. The expected claim frequency is therefore $2\times 0.1634859+0.3512505\times 0.8365141 = 0.6207978$.

[Technically, since we calculated the mean per policy-year, the per-policy year approach is more correct.]

Standard Questions

5. *Aggregate claims for a given insurance policy are modelled as following a log-normal, and experience indicates that $\sigma = 1$ is the correct value. The first 4 years of experience on this policy are:*

Policyholder	*Year 1*	*Year 2*	*Year 3*	*Year 4*
1	*970*	*221*	*703*	*366*
2	*282*	*392*	*2934*	*1372*
3	*213*	*631*	*339*	*551*
4	*54*	*2383*	*252*	*647*

(a) Estimate the EPV and VHM based on the MLE estimates for each μ. [For a log-normal the MLE is $\hat{\mu} = \frac{\sum_{i=1}^{n}\log(X_i)}{n}$.]

The MLE estimates are

$$\frac{\log(970)+\log(221)+\log(703)+\log(366)}{4}=6.183362$$
$$\frac{\log(282)+\log(392)+\log(2934)+\log(1372)}{4}=6.705329$$
$$\frac{\log(213)+\log(631)+\log(339)+\log(551)}{4}=5.986583$$
$$\frac{\log(54)+\log(2383)+\log(252)+\log(647)}{4}=5.941719$$

The mean of a log-normal is $e^{\mu+\frac{1}{2}\sigma^2}$, and the variance is $e^{2\mu+2\sigma^2}-e^{2\mu+\sigma^2}=e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$.
This gives that the EPV is $(e-1)\frac{e^{13.366724}+e^{14.410658}+e^{12.973166}+e^{12.883438}}{4}=1407332$. The mean is $\frac{e^{6.683362}+e^{7.205329}+e^{6.486583}+e^{6.441719}}{4}$ 857.3375 while the variance of estimated means is $\frac{(e^{6.683362}-857.3375)^2+(e^{7.205329}-857.3375)^2+(e^{6.486583}-857.3375)^2+(e^{6.441719}-857.3375)^2}{3}=$ 112008.9. For a log-normal distribution with known $\sigma=1$, the estimate $\hat{\mu}$ is normally distributed with mean μ and variance $\frac{sigma^2}{4}$ (since it is the mean of the logarithms of the observed values). The estimated mean of the log-normal distribution is $e^{\hat{\mu}+\frac{1}{2}}$, so the variance of this is $\mathbb{E}((e^{\hat{\mu}+\frac{1}{2}})^2)-\mathbb{E}(e^{\hat{\mu}+\frac{1}{2}})^2$. From the variance for a log-normal (or the mgf of the normal distribution), we have $\mathbb{E}((e^{\hat{\mu}+\frac{1}{2}})^2)=e^{2\mu+1+\frac{1}{2}}$ and $\mathbb{E}(e^{\hat{\mu}+\frac{1}{2}})^2=e^{2\left(\mu+\frac{1}{2}+\frac{1}{8}\right)}=e^{2\mu+\frac{5}{4}}$. The variance is therefore $e^{2\mu}\left(e^{\frac{3}{2}}-e^{\frac{5}{4}}\right)$. In particular, the estimated variances for the means for each policyholder are:

Policyholder 1	$e^{2\times 6.183362}\left(e^{\frac{3}{2}}-e^{\frac{5}{4}}\right)$	$=232822.7$
Policyholder 2	$e^{2\times 6.705329}\left(e^{\frac{3}{2}}-e^{\frac{5}{4}}\right)$	$=661302.4$
Policyholder 3	$e^{2\times 5.986583}\left(e^{\frac{3}{2}}-e^{\frac{5}{4}}\right)$	$=157074.3$
Policyholder 4	$e^{2\times 5.941719}\left(e^{\frac{3}{2}}-e^{\frac{5}{4}}\right)$	$=143594.2$

The VHM is therefore $112008.9-\left(\frac{232822.7+661302.4+157074.3+143594.2}{4}\right)=$

(b) Calculate the credibility premium for policyholder 3 in the next year.

The credibility of 4 years of experience is therefore $Z=\frac{4}{4+\frac{1407332}{112008.9}}=0.2414808$. The credibility premium is therefore $0.2414808\times e^{6.486583}+0.7585192\times 857.3375=\$808.79$.

6. *Claim frequency in a year for an individual follows a Poisson with parameter Λt where Λ is the individual's risk factor and t is the individual's exposure in that year. An insurance company collects the following data:*

	Year 1		*Year 2*		*Year 3*		*Year 4*	
Policyholder	*Exp*	*claims*	*Exp*	*claims*	*Exp*	*claims*	*Exp*	*claims*
1	*45*	*12*	*10*	*6*	*45*	*14*	*14*	*2*
2	*27*	*0*	*12*	*0*	*74*	*0*	*27*	*0*
3	*10*	*9*	*293*	*149*	*14*	*6*	*13*	*5*
4	*10*	*0*	*14*	*3*	*17*	*2*	*6*	*2*

In year 5, policyholder 3 has 64 units of exposure. Calculate the credibility estimate for claim frequency for policyholder 3.

The total exposure is $45+10+45+14+27+12+74+27+10+293+14+13+10+14+17+6=631$ units.

The means for each individual are: $\frac{34}{114}=0.2982456$, $\frac{0}{140}=0$, $\frac{169}{330}=0.5121212$, and $\frac{7}{47}=0.1489362$. The average value of λ is therefore $\frac{0.2982456+0+0.5121212+0.1489362}{4}=0.2398258$, so this is the expected process variance, because the variance of a Poisson distribution is equal to the mean.

Suppose the hypothetical means are λ_1, λ_2, λ_3, and λ_4, and the overall mean is $\lambda=\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4}$. The variance of means is $\frac{(\lambda_1-\lambda)^2+(\lambda_2-\lambda)^2+(\lambda_3-\lambda)^2+(\lambda_4-\lambda)^2}{3}$, We estimate this as

$$\frac{(0.2982456-0.2398258)^2+(0-0.2398258)^2+(0.5121212-0.2398258)^2+(0.1489362-0.2398258)^2}{3}=0.04777833$$

However, this estimate also includes the variance of each estimate for λ_i. The variance if the total exposure is n_i is $\frac{\lambda_i}{\sqrt{n_i}}$, so the total variance due to process variance is

$$\frac{0.2982456}{16\sqrt{114}}+0+\frac{0.2982456}{16\sqrt{330}}+\frac{0.2982456}{16\sqrt{47}}=0.004865574$$

The VHM is therefore $0.04777833-0.004865574=0.04291276$.

The credibility for an individual with 330 units of exposure is therefore $\frac{330}{330+\frac{0.2398258}{0.04291276}}=0.9833466$

The credibility estimate for λ_3 is therefore $0.9833466\times 0.5121212+0.0166534\times 0.2398258=0.5075866$.