ACSC/STAT 4703, Actuarial Models II
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Midterm Examination
Model Solutions

Each part question (a, b, c, etc.) is worth 1 mark. You should have been provided with a formula sheet. **No other notes are permitted**. Scientific calculators are permitted, but not graphical calculators.

Here are some values of the Gamma distribution function with $\theta = 1$ that may be needed for this examination:

x	α	$S(x)$
4087.353	3754.498	6.287484×10^{-8}
4087.353	3755.498	6.864219×10^{-8}
540.54	455	0.00007273623
199.0521	158.291	0.001250645
199.0521	157.291	0.0009683224
204.3453	158.291	0.9996398
204.3453	157.291	0.9997278

1. *Claim frequency follows a Poisson distribution with $\lambda = 3.4$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
0	*0*
1	*0.470*
2	*0.350*

Use the recursive method to calculate the exact probability that aggregate claims are at least 3.

For the Poisson distribution we have $a = 0, b = 3.4$. Substituting this into the recursive formula gives

$$f_S(x) = \sum_{y=1}^{x} 3.4\frac{y}{x} f_C(y) f_S(x-y)$$

Since severity is never 0, the probability that the aggregate loss is zero is the probability that the number of claims is zero, which is $e^{-3.4} = 0.03337327$.

We then use the recurrence to get

$$f_S(1) = 3.4 \times 0.470e^{-3.4} = 0.05333049$$

$$f_S(2) = 3.4\frac{1}{2} \times 0.470 \times 0.05333049 + 3.4 \times 0.350e^{-3.4} = 0.08232525$$

The probability that the aggregate loss exceeds 3 is therefore

$$1 - 0.03337327 - 0.05333049 - 0.08232525 = 0.8310$$

2. *An insurance company has the following portfolio of insurance policies:*

Type of policy	*Number*	*Probability claim*	*mean of claim*	*standard deviation*
Collision insurance	*2700*	*0.11*	*2400*	*3800*
Comprehensive insurance	*1500*	*0.07*	*4500*	*3300*

Calculate the cost of reinsuring losses above $1,500,000, if the reinsurance premium is the expected claim payment on the reinsurance policy, using a gamma approximation for the aggregate losses on this portfolio.

The expected aggregate loss is

$$2700 \times 0.11 \times 2400 + 1500 \times 0.07 \times 4500 = 712800 + 472500 = 1,185,300$$

The variance of aggregate loss is

$$2700\times 0.11\times 0.89\times 2400^2+2700\times 0.11\times 3800^2+1500\times 0.07\times 0.93\times 4500^2+1500\times 0.07\times 3300^2 = 8932083300$$

The parameters α and θ of the gamma distribution therefore satisfy

$$\alpha\theta = 1185300$$
$$\alpha\theta^2 = 8932083300$$

Which gives $\theta = \frac{8932083300}{1185300} = 7535.715$ and $\alpha = \frac{1185300}{7535.715} = 157.291$.

The expected payment on the excess-of-loss insurance is therefore

$$\int_{1500000}^{\infty} (x - 1500000)\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx \int_{1500000}^{\infty} \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx - 1500000\int_{1500000}^{\infty} \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx$$
$$= \alpha\theta\int_{1500000}^{\infty} \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha+1)}\,dx - 1500000\int_{1500000}^{\infty} \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx$$

The first integral is the probability that a gamma distribution with $\alpha+1$ and $\theta = 1$ is more than $\frac{1500000}{\theta} = 199.0521$ and the second is the probability that a gamma distribution with α and $\theta = 1$ is more than $\frac{1500000}{\theta} = 199.0521$. From the table, these are 0.001250645 and 0.0009683224 respectively. The expected payment is therefore

$$1185300 \times 0.001250645 - 1500000 \times 0.0009683224 = \$29.91$$

3. *For the following dataset:*

    0.2 0.6 0.9 1.6 2.4 2.5 7.2 7.3 8.7 10.4

Calculate a Nelson-Åalen estimate for the probability that a random sample is more than 2.

The Nelson-Åalen estimate is $H(2) = \frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \frac{1}{7} = 0.4789683$. This gives the probability that a sample is more than 2 is $S(2) = e^{-H(2)} = e^{-0.4789683} = 0.6194221$.

4. *An insurance company collects the following data on 1000 insurance claims:*

Claim Amount	*Number of Claims*
Less than \$10,000	*459*
\$10,000–\$20,000	*279*
\$20,000–\$50,000	*168*
\$50,000–\$100,000	*72*
More than \$100,000	*22*

Using the ogive to estimate the empirical distribution, find the policy limit which would make the TVaR at the 95% level, of the policy equal to 85,000?

From the data, we have $F_n(50000) = 0.906$ and $F_n(100000) = 0.978$, so using the ogive, for values x between 50,000 and 100,000, we have $F_n(x) = 0.987\frac{x-50000}{50000} + 0.906\frac{100000-x}{50000}$. We want to solve

$$\begin{aligned}
F_n(x) &= 0.95 \\
0.987\frac{x - 50000}{50000} + 0.906\frac{100000 - x}{50000} &= 0.95 \\
0.081x + 90600 - 49350 &= 47500 \\
0.081x &= 6230 \\
x &= 76913.58
\end{aligned}$$

This gives that the VaR is \$76,913.58. The TVaR is the conditional expected value given that $x \geqslant 76913.58$. Setting this as 85000 gives

$$\begin{aligned}
\frac{\int_{76913.58}^{u} S_n(x)\,dx}{0.05} + 76913.58 &= 85000 \\
\int_{76913.58}^{u} S_n(x)\,dx &= 0.05 \times 8086.42 = 404.321
\end{aligned}$$

For $76913.58 \leqslant x \leqslant 100000$ we have $S_n(x) = \frac{8750-0.081x}{50000}$ so

$$\int_{76913.58}^{u} S_n(x)\,dx = \frac{8750}{50000}(u - 76913.58) - \frac{0.081}{100000}(u^2 - 76913.58^2)$$

We therefore need to solve

$$\begin{aligned}
\frac{8750}{50000}(u - 76913.58) - \frac{0.081}{100000}(u^2 - 76913.58^2) &= 404.321 \\
\frac{0.081}{100000}u^2 - \frac{8750}{50000}u + 404.321 + 76913.58\frac{8750}{50000} - 76913.58^2\frac{0.081}{100000} &= 0 \\
8.1 \times 10^{-7}u^2 - 0.175u + 404.321 + 13459.88 - 4791.716 &= 0 \\
8.1 \times 10^{-7}u^2 - 0.175u + 9072.485 &= 0
\end{aligned}$$

The solution to this is

$$\begin{aligned}
u &= \frac{0.175 \pm \sqrt{0.175^2 - 4 * 8.1 \times 10^{-7} \times 9072.485}}{8.1 \times 10^{-7}} \\
&= 86374.40
\end{aligned}$$

[The other solution is more than 100000, and involves extrapolating the linear form outside the interval so that the survival function becomes negative.]

5. *An insurance company collects the following claim data (in thousands):*

i	d_i	x_i	u_i	i	d_i	x_i	u_i
1	0.5	0.6	-	6	0.5	-	10
2	0.5	1.7	-	7	2.0	1.2	-
3	0.5	9.2		8	2.0	6.4	-
4	0.5	-	5	9	2.0	7.7	-
5	0.5	-	10	10	2.0	-	10

Using a Kaplan-Meier product-limit estimator:

(a) estimate the probability that a random loss exceeds 8

We construct the following table:

i	y_i	r_i	s_i	$S(y_i)$
1	0.6	6	1	$\frac{5}{6}$
2	1.2	5	1	$\frac{4}{6}$
3	1.7	4	1	$\frac{1}{2}$
4	6.4	6	1	$\frac{5}{12}$
5	7.7	5	1	$\frac{1}{3}$
6	9.2	4	1	$\frac{1}{4}$

So the probability of a random loss exceeding 8 is $\frac{1}{3}$.

(b) Use Greenwood's approximation to find a log-transformed 95% confidence interval for the probability that a random loss exceeds 8.

Using Greenwood's approximation, the variance of $\hat{S}(8)$ is

$$\left(\frac{1}{3}\right)^2\left(\frac{1}{5\times 6}+\frac{1}{4\times 5}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\frac{1}{4\times 5}\right)=\frac{1}{9}\times\frac{2+3+5+2+3}{60}=\frac{1}{36}$$

A log-transformed confidence interval is then $\left[\left(\frac{1}{3}\right)^U,\left(\frac{1}{3}\right)^{\frac{1}{U}}\right]$, where $U = e^{-\frac{1.96\sqrt{\frac{1}{36}}}{\left(\frac{1}{3}\right)\log\left(\frac{1}{3}\right)}} = 2.440089$

The confidence interval is therefore

$$\left[\left(\frac{1}{3}\right)^U,\left(\frac{1}{3}\right)^{\frac{1}{U}}\right]=[0.06851437, 0.6374786]$$

6. *An insurance company records the following data in a mortality study:*

Age	*no. at start*	*no. enter*	*no. die*	*no exit*
64	*743*	*232*	*44*	*126*
65	*795*	*206*	*52*	*113*
66	*836*	*216*	*73*	*134*
67	*845*	*195*	*74*	*131*

Estimate the probability of an individual currently aged exactly 66 dying within the next year using the actuarial exposure method. [You may assume all events happen in the middle of the year.]

For an individual aged 66, the actuarial exposure is $836+\frac{216-134}{2}=877$. The probability of dying is therefore $\frac{73}{877}=0.08323831$.

7. *An insurance company models number of claims an individual makes in a year as following a Poisson distribution with Λ an unknown parameter with prior distribution a gamma distribution with $\alpha = 2$ and $\theta = 0.11$.*

The company observes 1000 policies for a year and finds there have been 293 claims. Calculate the predictive probability that an individual makes no claims next year.

The likelihood that 1000 individuals make 293 claims is proportional to $\lambda^{293}e^{-1000\lambda}$, so the posterior density is proportional to $\lambda^{293}e^{-1000\lambda}\lambda^{2-1}e^{-\frac{\lambda}{0.11}} = \lambda^{294}e^{-1009.090909\lambda}$ Which is the density of a gamma distribution with $\alpha = 295$ and $\theta = \frac{1}{1009.0909}$. For a fixed value of λ, the probability an individual makes no claims is $e^{-\lambda}$. We therefore want to calculate $\mathbb{E}\left(e^{-\Lambda}\right) = M_\lambda(-1)$, where Λ follows a gamma distribution with $\alpha = 295$ and $\theta = \frac{1}{1009.0909}$. We have $M_\Lambda(t) = (1-\theta t)^{-\alpha}$. Substituting $t = -1$, we get that the predictive probability is $\left(1-\frac{-1}{1009.0909}\right)^{-295} = 0.746621$.