ACSC/STAT 4703, Actuarial Models II
Fall 2016
Toby Kenney
Homework Sheet 2
Model Solutions

Basic Questions

1. *An insurance company has the following portfolio of home insurance policies:*

Type of home	*Number*	*Probability of claim*	*mean claim*	*standard deviation*
Small home	*1300*	*0.10*	*$35,000*	*$29,000*
Medium home	*1200*	*0.09*	*$51,000*	*$39,000*
Large home	*400*	*0.07*	*$200,000*	*$88,000*

Calculate the cost of reinsuring losses above $10,000,000, if the loading on the reinsurance premium is one standard deviation above the expected claim payment on the reinsurance policy using a gamma approximation for the aggregate losses on this portfolio.

The mean aggregate loss is

$$1300\times 0.1\times 35000+1200\times 0.09\times 51000+400\times 0.07\times 200000 = \$15,658,000$$

The variance of aggregate loss is

$$\begin{aligned}&1300(0.1\times 29000^2+0.1\times 0.9\times 35000^2)+1200(0.09\times 0.91\times 51000^2+0.09\times 39000^2)\\&\quad+400(0.07\times 0.93\times 200000^2+0.07\times 88000^2)\\&=\$1,930,981,280,000\end{aligned}$$

A gamma distribution with parameters α and θ has mean $\alpha\theta$ and variance $\alpha\theta^2$, so to match the moments calculated, we need $\theta=\frac{1930981280000}{15658000}=123322.3$ and $\alpha=\frac{15658000^2}{1930981280000}=126.9681$.

Recall that the expected excess-of-loss variable for a gamma distribution is given by

$$\int_d^\infty (x-d)\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx=\alpha\theta\int_d^\infty \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha+1)}\,dx-d\int_d^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx$$

The first integral above is the probability that a gamma distribution with parameters $\alpha+1$ and θ is more than d, while the second is the probability that a gamma distribution with parameters α and θ exceeds d.

`R` code:

```
alp<-126.9681
tht<-123322.3
alp*tht*pgamma(20000000,alp+1,scale=tht,lower.tail=F)\
    -20000000*pgamma(20000000,alp,scale=tht,lower.tail=F)
```

This gives the mean excess-of-loss as \$885.3063

To get the variance of excess-of-loss, we calculate

$$\begin{aligned}\mathbb{E}((X-d)_+{}^2) &= \int_d^\infty (x-d)^2 \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx \\ &= \alpha(\alpha+1)\theta^2\int_d^\infty \frac{x^{\alpha+1}e^{-\frac{x}{\theta}}}{\theta^{\alpha+2}\Gamma(\alpha+2)}\,dx - 2d\alpha\theta\int_d^\infty \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha+1)}\,dx + d^2\int_d^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)}\,dx\end{aligned}$$

The integrals give the probability of gamma random variables exceeding d.

`R` code:

```
alp<-126.9681
tht<-123322.3
alp*tht*pgamma(20000000,alp+1,scale=tht,lower.tail=F)\
    -20000000*pgamma(20000000,alp,scale=tht,lower.tail=F)
```

this gives $\mathbb{E}((X-d)_+{}^2) = 808018042$. The variance of the excess-of-loss is therefore $808018042 - 885.3063^2 = 807234275$, and the standard deviation is $\sqrt{807234275} = 28411.87$.

The premium for the excess-of-loss insurance is therefore $885.3063+28411.87 =$ \$29,297.17.

2. *For the following dataset*

   0.9 1.3 2.6 0.6 1.5 3.8 2.1 0.6 1.6 0.2 1.5 0.7 2.2
   2.1 1.1 2.2 0.1 1.9 1.4 1.2 1.8 1.3 2.6

   Use the Emprirical distribution function to calculate the cumulative hazard rate $H(2.5)$.

   We first sort the sample

   ```
   0.1 0.2 0.6 0.6 0.7 0.9 1.1 1.2 1.3 1.3 1.4 1.5 1.5
   1.6 1.8 1.9 2.1 2.1 2.2 2.2 2.6 2.6 3.8
   ```

   We therefore see that there are 3 observations out of 23 larger than 2.5. This gives $S(2.5) = \frac{3}{23}$, so the cumulative hazard rate is $H(2.3) = -\log\left(\frac{3}{23}\right) = 2.036882$.

3. *For the sample from Question 2, calculate a Nelson-Åalen estimate for the probability that a random sample is more than 1.85.*

   The Nelson-Åalen estimate for $H(1.85)$ is

   $$\frac{1}{23}+\frac{1}{22}+\frac{2}{21}+\frac{1}{19}+\frac{1}{18}+\frac{1}{17}+\frac{1}{16}+\frac{2}{15}+\frac{1}{13}+\frac{2}{12}+\frac{1}{10}+\frac{1}{9}=1.001716$$

   We then calculate

   $$S(1.85)=e^{-1.001716}=0.3672487$$

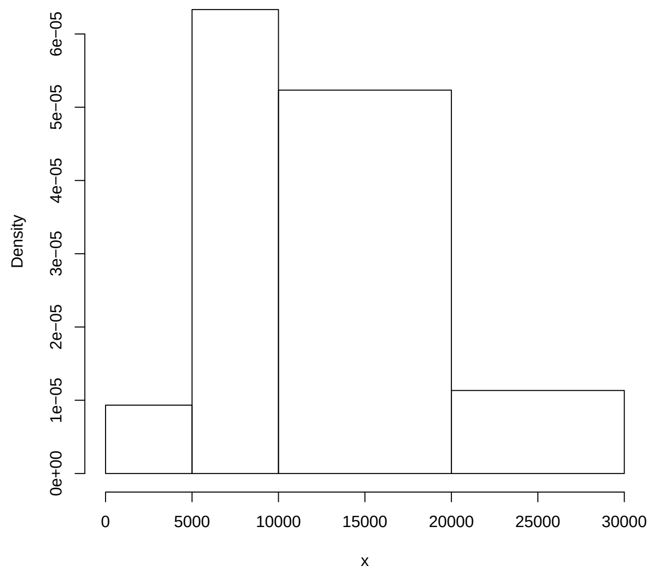
4. *Draw a histogram of the following distribution:*

Claim Amount	*Number of Claims*
Less than \$5,000	*14*
\$5,000–\$10,000	*95*
\$10,000–\$20,000	*157*
\$20,000–\$30,000	*34*

   The total number of claims is 300, so the densities are:

   $$\frac{14}{300\times 5000}=0.00000933333$$
   $$\frac{95}{300\times 5000}=0.00006333333$$
   $$\frac{157}{300\times 10000}=0.00005233333$$
   $$\frac{34}{300\times 10000}=0.00001133333$$

   This gives the following histogram:



Standard Questions

5. *An insurance company insures 3 types of drivers with the following characteristics*

Type of driver	*Number of claim*	*Probability claim*	*mean deviation*	*standard*
Good driver	*200*	*0.01*	*\$3,500*	*\$1,300*
Average driver	*1,500*	*0.02*	*\$3,800*	*\$1,700*
Bad driver	*550*	*0.11*	*\$4,700*	*\$2,800*

An insurance company sets the premium for each policy at 0.5 *standard deviations above the mean annual claim (assume no more than one claim per year). The company models aggregate losses using a Pareto distribution with the first two moments matching the true distribution. The insurance company uses excess of loss insurance to ensure it cannot make a loss on this portfolio (ignoring administrative costs). The reinsurance company charges a premium at the mean plus one standard deviation of the payment on the excess loss. Calculate the premium for the excess-of-loss reinsurance. [Remember to include this premium in the insurance company's loss, when setting the level for the excess-of-loss reinsurance.]*

The variances of the loss for the three types of drivers are

$$
\begin{aligned}
0.01\times 1300^2+0.01\times 0.99\times 3500^2 &= 138175\\
0.02\times 1700^2+0.02\times 0.98\times 3800^2 &= 340824\\
0.11\times 2800^2+0.11\times 0.89\times 4700^2 &= 3025011
\end{aligned}
$$

The corresponding standard deviations and premiums are

Type of driver	Number of claim	variance deviation	standard	premium
Good driver	200	138175	371.7190	$ 220.86
Average driver	1,500	340824	583.8013	$ 367.90
Bad driver	550	3025011	1739.2559	$1,386.63

The mean aggregate loss for the portfolio is

$$200\times 0.01\times 3500+1500\times 0.02\times 3800+550\times 0.11\times 4700=405,350$$

The variance of aggregate loss for the portfolio is

$$
\begin{aligned}
&200(0.01\times 1300^2+0.01\times 0.99\times 3500^2)+1500(0.02\times 1700^2+0.02\times 0.98\times 3800^2)+\\
&\qquad 550(0.11\times 2800^2+0.11\times 0.89\times 4700^2)=2,202,627,050
\end{aligned}
$$

Using a Pareto distribution, the mean is $\frac{\theta}{\alpha-1}$ and the variance is $\frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$. We therefore need to solve

$$
\begin{aligned}
\frac{\theta^2}{(\alpha-1)^2(\alpha-2)} &= 2202627050\\
\frac{\theta}{\alpha-1} &= 405350\\
\alpha-2 = \frac{405350^2}{2202627050} &= 74.59666\\
\theta = 405350\times 75.59666 &= 30643106
\end{aligned}
$$

The total of all premiums received is $200\times 220.86+1500\times 367.90+550\times 1386.63=\$1,358,668$.

The insurance company therefore wants to insure excess-of-loss at a level u, with premium P, so that $u+P=1358668$. For the excess-of-loss insurance at level u, the expected excess-of-loss is

$$\int_u^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx=\int_{u+\theta}^\infty \theta^\alpha z^{-\alpha}\,dz=\theta^\alpha\left[\frac{z^{1-\alpha}}{1-\alpha}\right]_{u+\theta}^\infty=\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}$$

The expected square excess-of-loss is

$$\begin{aligned}\int_u^\infty \frac{(x-u)^2\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}\,dx &= \alpha\theta^\alpha\int_{u+\theta}^\infty z^{-(\alpha+1)}(z-u-\theta)^2\,dz\\
&= \alpha\theta^\alpha\left(\int_{u+\theta}^\infty z^{-(\alpha-1)}+(u+\theta)^2z^{-(\alpha+1)}-2(u+\theta)z^{-\alpha}\,dz\right)\\
&= \frac{\alpha\theta^\alpha}{(u+\theta)^{\alpha-2}}\left(\frac{1}{\alpha-2}+\frac{1}{\alpha}-\frac{2}{\alpha-1}\right)\\
&= \frac{\theta^\alpha}{(\alpha-1)(\alpha-2)(u+\theta)^{\alpha-2}}\end{aligned}$$

The variance of the excess-of-loss is therefore

$$\frac{\theta^\alpha}{(\alpha-1)(\alpha-2)(u+\theta)^{\alpha-2}}-\left(\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\right)^2$$

The premium for the excess-of-loss insurance is

$$P=\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}+\sqrt{\frac{\theta^\alpha}{(\alpha-1)(\alpha-2)(u+\theta)^{\alpha-2}}-\left(\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\right)^2}$$

We want to solve $u+P=1358668$

$$\begin{aligned}u+\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}+\sqrt{\frac{\theta^\alpha}{(\alpha-1)(\alpha-2)(u+\theta)^{\alpha-2}}-\left(\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\right)^2}&=1358668\\
\sqrt{\frac{\theta^\alpha}{(\alpha-1)(\alpha-2)(u+\theta)^{\alpha-2}}-\left(\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\right)^2}&=1358668-u-\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\\
\sqrt{\frac{\theta^\alpha}{(\alpha-1)(\alpha-2)(u+\theta)^{\alpha-2}}-\left(\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\right)^2}&=1358668+\theta-(u+\theta)-\frac{\theta^\alpha}{(\alpha-1)(u+\theta)^{\alpha-1}}\end{aligned}$$

Substituting $z=\frac{u+\theta}{\theta}$, this expression becomes

$$\begin{aligned}
\sqrt{\frac{a}{z^{\alpha-2}}-\left(\frac{b}{z^{\alpha-1}}\right)^2} &= c-\theta z-\frac{b}{z^{\alpha-1}}\\
\frac{a}{z^{\alpha-2}}-\left(\frac{b}{z^{\alpha-1}}\right)^2 &= c^2+\theta^2z^2+\left(\frac{b}{z^{\alpha-1}}\right)^2-2c\theta z-\frac{2bc}{z^{\alpha-1}}-\frac{2b\theta}{z^{\alpha-2}} \text{ where}\\
\frac{a+2b\theta}{z^{\alpha-2}} &= c^2+\theta^2z^2+\frac{2b^2}{z^{2\alpha-2}}-2c\theta z-\frac{2bc}{z^{\alpha-1}}
\end{aligned}$$

$$\begin{aligned}
a &= \frac{\theta^2}{(\alpha-1)(\alpha-2)} = 166511249550\\
b &= \frac{\theta}{(\alpha-1)} = 40535\\
c &= 1358668+\theta = 32001774
\end{aligned}$$

We solve this numerically to get $z = 1.0405663039422$, which gives $u = \theta(z-1) = \$1,243,078$

6. *An insurance company collects the following data on insurance claims:*

Claim Amount	*Number of Policies*
Less than \$10,000	*41*
\$10,000–\$20,000	*477*
\$20,000–\$50,000	*2006*
\$50,000–\$100,000	*470*
More than \$100,000	*6*
Total	*3000*

Using the ogive as an estimate for the empirical distribution function:

(a) estimate the VaR of the severity distribution at the 95% level.

We are trying to solve $F(x) = 0.95$. The ogive is found by linearly interpolating between known values of the empirical distribution function. We have $F(50000) = \frac{2524}{3000}$ and $F(100000) = \frac{2994}{3000}$. We therefore get

$$F(50000a+100000(1-a)) = \frac{2525a+2994(1-a)}{3000}$$

We are therefore aiming to solve

$$\begin{aligned}
\frac{2525a + 2994(1-a)}{3000} &= 0.95 = \frac{2850}{3000} \\
2525a + 2994(1-a)2850 \\
2994 - 469a &= 2850 \\
469a &= 144 \\
a &= \frac{144}{469} = 0.3070362
\end{aligned}$$

which gives

$$50000a + 100000(1-a) = 50000 \times 0.3070362 + 100000 \times (1 - 0.3070362) = 65351.81$$

(b) estimate the TVaR at the 95% level assuming the policy has a policy limit of $100,000.

We are aiming to estimate the expected claim conditional on the claim exceeding $65,351.81. Conditional on this, the probability that the loss exceeds $100,000 is $\frac{\left(\frac{6}{3000}\right)}{0.05} = 0.1$. Otherwise, using the ogive, the loss is uniformly distributed between $65,351.81 and $100,000. The expected loss under this uniform distribution is $\frac{65351.81+100000}{2} = \$82,675.91$, and the overall expected loss is $82675.91 \times 0.9 + 100000 \times 0.1 = \$84,408.32$.