ACSC/STAT 4703, Actuarial Models II
Fall 2017
Toby Kenney
Homework Sheet 4
Model Solutions

Basic Questions

1. *An insurance company sells car insurance. It estimates that the standard deviation of the aggregate annual claim is \$3,691 and the mean is \$725.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.01$, $p = 0.95$.)

   For n years history, the variance of their average annual claim is $\frac{3691^2}{n}$. The mean is 725, so for a relative error of 1%, we want the probability of the individual's mean annual claim amount being within 1% of the truth (or within \$7.25 of the truth) to be 95%. That is, we want

$$\Phi\left(\frac{7.25\sqrt{n}}{3691}\right) = 0.975$$
$$\frac{7.25\sqrt{n}}{3691} = 1.96$$
$$n = \left(\frac{3691 \times 1.96}{7.25}\right)^2 = 995690.170932$$

   So 995691 years are needed.

   The standard premium for this policy is \$725. An individual has claimed a total of \$3,300 in the last 10 years.

   (b) What is the Credibility premium for this individual, using limited fluctuation credibility?

   The credibility we assign to this individual's experience is $Z = \sqrt{\frac{10}{995690.170932}} =$ 0.00316911420446. The credibility premium is therefore $725 \times 0.99683089 + 330 \times 0.00316911 = \$723.75$.

2. *A car insurance company classifies drivers as good or bad. Annual claims from good drivers follow a gamma distribution with $\alpha = 4$ and $\theta = 200$. Annual claims from bad drivers follow a Pareto distribution with shape $\alpha = 5$ and $\theta = 6000$. 75% of individuals are good drivers.*

   (a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen driver.

The expectation of aggregate annual claims from a good driver is $\alpha\theta = 800$. The expectation of aggregate annual claims from a bad driver is $\frac{\theta}{\alpha-1} = \$1,500$. The expected aggregate annual claims from a random driver are therefore $0.75 \times 800 + 0.25 \times 1500 = \$9,75$. The variance of annual aggregate claims from good drivers is $\alpha\theta^2 = 160000$, and for bad drivers it is $\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} = 3750000$. The variance for a random driver is therefore

$$0.75 \times 160000 + 0.25 \times 3750000 + 0.75 \times 0.25 \times (1500 - 800)^2 = 1,149,375$$

(b) Given that a driver's annual claims over the past 3 years are \$1,000, \$600 and \$800, what are the expectation and variance of the driver's claims next year?

If the driver is good, the likelihood of these claims is

$$\frac{(1000)^3(600)^3(800)^3e^{-\frac{2400}{200}}}{200^{12}\Gamma(4)^3} = 7.680265 \times 10^{-10}$$

If the driver is bad, the likelihood is

$$\frac{5^3(6000)^{15}}{(6000+1000)^6(6000+600)^6(6000+800)^6} = 6.1132680551 \times 10^{-11}$$

The posterior probability that the driver is bad is therefore $\frac{0.25\times6.1132680551\times10^{-11}}{0.25\times6.1132680551\times10^{-11}+0.75\times7.680265\times10^{-10}} = 0.025846594675$. The expected claim for this driver next year is $0.025846594675 \times 1500 + (1 - 0.025846594675) \times 800 = 818.09$. The variance is

$$0.025846594675\times3750000+(1-0.025846594675)\times160000+0.025846594675\times(1-0.025846594675)\times700^2 = 265126.76$$

3. *The number of claims made by an individual in a year follows a Poisson distribution with mean Λ, where the value of Λ follows a Pareto distribution with $\alpha = 4.6$ and $\theta = 0.24$. Given that an individual has made three claims in the past 7 years, what is the expected number of claims made in the next year?*

   The likelihood of the individual making 3 claims in the past 7 years is proportional to $L(\lambda) = e^{-7\lambda}\lambda^7$. The prior distribution of λ is proportional to $f(\lambda) = \frac{4.6(0.24)^{4.6}}{(0.24+\lambda)^{5.6}}$. The posterior distribution is therefore proportional to

$$f_{\Lambda|X}(\lambda) = e^{-7\lambda}\frac{\lambda^7}{(0.24+\lambda)^{5.6}}$$

   Numerically integrating this, we get

$$\int_0^\infty e^{-7\lambda}\frac{\lambda^7}{(0.24+\lambda)^{5.6}}\,d\lambda = 0.0007387927$$
$$\int_0^\infty e^{-7\lambda}\frac{\lambda^8}{(0.24+\lambda)^{5.6}}\,d\lambda = 0.000439192$$
$$\mathbb{E}(\Lambda|X) = \frac{0.000439192}{0.0007387927} = 0.5944726$$

[By substituting $M = \Lambda + 0.24$, we see that M follows a linear combination of truncated Gamma distributions. In particular $f_M(m) \propto e^{-7m}\sum_{i=0}^{7}\binom{7}{i}0.24^{7-i}m^{i-5.6}$. We can then express the posterior mean in terms of truncated Gamma distributions and use integration by parts, however this does not simplify the computation.]

Standard Questions

4. *For a certain insurance policy, the book premium is based on average claim frequency of 0.3 claims per year, and average claim severity of \$4,030. A particular group has made 130 claims from 987 policies in the last year. The average claim severity was \$7,414. Estimate the credibility premium for this group using limited fluctuation credibility if the standard for full credibility is:*

   (a) 203 claims for claim frequency, 740 claims for severity.

   For claim frequency, the credibility is $\sqrt{\frac{130}{203}} = 0.8002463$, so the credibility estimate is $0.8002463 \times \frac{130}{987} + 0.1997537 \times 0.3 = 0.165328358227$. For claim severity the credibility is $Z = \sqrt{\frac{130}{740}} = 0.4191368$, so the credibility estimate is $0.4191368 \times 7417 + 0.5808632 \times 4030 = 5449.6163416$. The credibility premium is therefore $0.165328358227 \times 5449.6163416 = \$900.98$.

   (b) 1406 policies for claim frequency, 740 claims for severity.

   For claim frequency, the credibility is $\sqrt{\frac{987}{1406}} = 0.8378493$, so the credibility estimate is $0.8378493 \times \frac{130}{987} + 0.1621507 \times 0.3 = 0.159000234316$. For claim severity the credibility is $Z = \sqrt{\frac{130}{740}} = 0.4191368$, so the credibility estimate is $0.4191368 \times 7417 + 0.5808632 \times 4030 = 5449.6163416$. The credibility premium is therefore $0.159000234316 \times 5449.6163416 = \$866.49$.

   (c) 1721 policies for aggregate claims.

   The average loss per policy last year was $\frac{130\times7417}{987} = 976.909827761$. The credibility is $\sqrt{\frac{987}{1721}} = 0.757300321454$, so the credibility premium is $0.757300321454 \times 976.909827761 + 0.242699678546 \times 0.3 \times 4030 = \$1033.24$.

5. *An insurance company has 3 years of past history on a driver, denoted X_1, X_2, X_3, X_4. It uses a formula $\hat{X}_5 = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4$ to calculate the credibility premium in the fourth year. It has the following information on the driver:*

   - *In year 1, the expected aggregate claim was $2,000.*
   - *Expected aggregate claims increase by 5% per year.*
   - *The coefficient of variation of the aggregate claims is 0.7 in every year.*
   - *The correlation (recall $Corr(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$) between aggregate claims in years i and j is $e^{-|i-j|}$.*

   Find a set of equations which can determine the values of α_0, α_1, α_2, α_3 and α_4. [You do not need to solve these equations.]

   Recall the standard credibility equations:

$$\mathbb{E}(X_5) = \alpha_0 + \sum_{i=1}^{4} \alpha_i \mathbb{E}(X_i)$$
$$\mathrm{Cov}(X_5, X_j) = \sum_{i=1}^{4} \alpha_i \,\mathrm{Cov}(X_i, X_j)$$

   In particular, we are given that

$$\begin{aligned}
\mathbb{E}(X_i) &= 2000(1.05)^{i-1} \\
\mathrm{Var}(X_i) &= 0.49\,(\mathbb{E}(X_i))^2 = 0.49 \times 2000^2 (1.05)^{2i-2} \\
\mathrm{Cov}(X_i, X_j) &= e^{-|i-j|} \sqrt{\mathrm{Var}(X_i)\,\mathrm{Var}(X_j)} \\
&= e^{-|i-j|} 0.49 \times 2000^2 (1.05)^{i+j-2}
\end{aligned}$$

   Plugging this into the equations gives

$$
\begin{aligned}
2000(1.05)^4 &= \alpha_0 + 2000\sum_{i=1}^{4}(1.05)^{i-1}\alpha_i \\
0.49 \times 2000^2(1.05)^{j+3}e^{j-5} &= \sum_{i=1}^{4}\alpha_i 0.49 \times 2000^2(1.05)^{j+i-2}e^{-|i-j|} \\
e^{j-5} &= \sum_{i=1}^{4}(1.05)^{i-5}e^{-|i-j|}\alpha_i
\end{aligned}
$$

$$
\begin{aligned}
2000(1.05)^4 &= \alpha_0 + 2000(\alpha_1 + 1.05\alpha_2 + 1.1025\alpha_3 + (1.05)^3\alpha_4) \\
e^{-4} &= (1.05)^{-4}\alpha_1 + (1.05)^{-3}e^{-1}\alpha_2 + (1.05)^{-2}e^{-2}\alpha_3 + (1.05)^{-1}e^{-3}\alpha_4 \\
e^{-3} &= (1.05)^{-4}e^{-1}\alpha_1 + (1.05)^{-3}\alpha_2 + (1.05)^{-2}e^{-1}\alpha_3 + (1.05)^{-1}e^{-2}\alpha_4 \\
e^{-2} &= (1.05)^{-4}e^{-2}\alpha_1 + (1.05)^{-3}e^{-1}\alpha_2 + (1.05)^{-2}\alpha_3 + (1.05)^{-1}e^{-1}\alpha_4 \\
e^{-1} &= (1.05)^{-4}e^{-3}\alpha_1 + (1.05)^{-3}e^{-2}\alpha_2 + (1.05)^{-2}e^{-1}\alpha_3 + (1.05)^{-1}\alpha_4
\end{aligned}
$$

[We can solve these relatively easily - subtracting e times the last equation from the second last gives

$$
\begin{aligned}
e^{-2} - 1 &= 1.05^{-1}(e^{-1} - e)\alpha_4 \\
\alpha_4 &= 1.05e^{-1}
\end{aligned}
$$

Then we subtract e times the third equation from the second to get

$$
\begin{aligned}
e^{-3} - e^{-1} &= 1.05^{-2}(e^{-1} - e)\alpha_3 + e^{-3} - e^{-1} \\
\alpha_3 &= 0
\end{aligned}
$$

similarly, $\alpha_2 = 0$, $\alpha_1 = 0$ and $\alpha_0 = 2100(1 - e^{-1})$.]