ACSC/STAT 4703, Actuarial Models II
Further Probability with Applications to Actuarial Science

Fall 2018

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In Class Examples

Advantages of Modelling Number of Losses and Severities Separately

- Dealing with changes to exposure (e.g. number of policies)
- Dealing with inflation
- Dealing with changes to individual policies
- Understanding the impact of changing deductibles on claim frequencies.
- Combining data with a range of different deductibles and limits can give a better picture of the loss distribution.
- Consistency between models of non-covered losses to insureds, claims to insurers, and claims to reinsurers.
- The effect of the shapes of separate distributions of number and severity give an indicator of how each influences the overall aggregate loss.

Model Choices

Practical Considerations

- Scale parameters for severity allow for change of currency or inflation.
- For frequency, models with pgf $P(z; \alpha) = Q(z)^\alpha$ can deal with changes to number of policies sold, or time period.
- Modification at zero prevents infinite divisibility. However, modification at zero may still be appropriate.

Question 1

Which discrete distributions satisfy

$$P(z;\alpha) = Q(z)^{\alpha}$$

for some parameter α?

Question 2

Calculate the first three moments of a compound model.

Question 3

When an individual is admitted to hospital, the distribution of charges incurred are as described in the following table:

charge	mean	standard deviation
Room	1000	500
other	500	300

The covariance between room charges and other charges is 100,000. An insurer issues a policy which reimburses 100% for room charges and 80% for other charges. The number of hospital admissions has a Poisson distribution with parameter 4. Determine the mean and standard deviation for the insurer's payout on the policy.

Question 4

An individual loss distribution is normal with mean 100 and standard deviation 35. The total number of losses N has the following distribution:

n	$P(N = n)$
0	0.4
1	0.3
2	0.2
3	0.1

What is the probability that the aggregate losses exceed 130?

Question 5

Aggregate payments have a compound distribution. The frequency distribution is negative binomial with $r = 16$, $\beta = 6$. The severity distribution is uniform on the interval $(0, 8)$. Using a normal approximation, determine the premium such that there is a 5% probabilty that aggregate payments exceed the premium.

Question 6

For a group health contract, aggregate claims are assumed to have an exponential distribution with mean θ estimated by the group underwriter. Aggregate stop-loss insurance for total claims in excess of 125% of the expected claims, is provided for a premium of twice the expected stop-loss claims. It is discovered that the expected total claims value used was 10% too low. What is the loading percentage on the stop-loss policy under the true distribution?

Question 7

Calculate the probability density function of the aggregate loss distribution if claim frequency follows a negative binomial distribution with $r = 2$ and severity follows an exponential distribution.

Question 8

An insurance company models the number of claims it receives as a negative binomial distribution with parameters $r = 15$ and $\beta = 2.4$. The severity of each claim follows an exponential distribution with mean \$3,000. What is the net-premium for stop-loss insurance with a deductible of \$500,000?

Question 9

An insurance company offers group life insurance policies to three different companies. For the first company, the number of claims is a Poisson distribution with parameter $\lambda = 0.4$, and claim severity a gamma distribution with $\theta = 30,000$ and $\alpha = 3$. For the second company, the number of claims is a Poisson distribution with parameter $\lambda = 3.6$ and the severity follows a gamma distribution with $\theta = 200,000$ and $\alpha = 1.4$. For the third company, the number of claims follows a Poisson distribution with $\lambda = 85$ and claim severity follows a gamma distribution with $\theta = 45,000$ and $\alpha = 2.2$. What is the probability that the aggregate claims from all these policies exceed 10,000,000?

Question 10

Suppose that the total number of claims follows a negative binomial distribution with $r = 2$ and $\beta = 3$. Suppose that the severity of each claim (in thousands of dollars) follows a zero-truncated ETNB distribution with $r = -0.6$ and $\beta = 7$. What is the probability that the aggregate loss is at most 3?

The Recursive Method

Theorem

Suppose the severity distribution is a discrete distribution with probability function $f_X(x)$ for $x = 0, 1, \ldots, m$ (m could be infinite) and the frequency distribution is a member of the $(a, b, 1)$ class with probabilities $p_k, k = 0, 1, 2, \ldots$ satisfying $p_k = \left(a + \frac{b}{k}\right) p_{k-1}$ for all $k \geqslant 2$.
Then the aggregate loss distribution is given by

$$f_S(x) = \frac{(p_1 - (a+b)p_0)f_X(x) + \sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - af_X(0)}$$

Question 11

Let the number of claims follow a Poisson distibution with $\lambda = 2.4$ and the severity of each claim follow a negative binomial distribution with $r = 10$ and $\beta = 2.3$. What is the probability that the aggregate loss is at most 3?

Question 12

An insurance company offers car insurance. The number of losses a driver experiences in a year follows a negative binomial random variable with $r = 0.2$ and $\beta = 0.6$. The size of each loss (in hundreds of dollars) is modelled as following a zero-truncated ETNB distribution with $r = -0.6$ and $\beta = 3$. The policy has a deductible of \$1,000 per loss. What is the probability that the company has to pay out at least \$400 in a single year to a driver under such a policy?

Question 13

The number of claims an insurance company receives is modelled as a compound Poisson distribution with parameter $\lambda = 6$ for the primary distribution and $\lambda = 0.1$ for the secondary distribution. Claim severity (in thousands of dollars) is modelled as following a zero-truncated logarithmic distribution with parameter $\beta = 4$. What is the probability that the total amount claimed is more than $3,000.

Question 14

The number of claims an insurance company receives is modelled as a Poisson distribution with parameter $\lambda = 96$. The size of each claim is modelled as a zero-truncated negative binomial distribution with $r = 4$ and $\beta = 2.2$. Calculate the approximated distribution of the aggregate claims:

(a) By starting the recursion at a value of k six standard deviations below the mean.

(b) By solving for a rescaled Poisson distribution with $\lambda = 12$ and convolving the solution up to 96.

Answer to Question 14

R-Code for (a)

```
ans<-1
ans<-as.vector(ans)
for(n in 2:2000){
  temp<-0
  for(i in 1:(n-1)){%
    temp<-temp+16*i*(i+1)*(i+2)*(i+3)/(n+240)*0.6875^i*
        0.3125^4*ans[n-i]/(1-0.3125^4)
  }
  ans<-c(ans,temp)
}
```

Answer to Question 14

R-Code for (b)

```
ConvolveSelf<-function(n){
  convolution<-vector("numeric",2*length(n))
  for(i in 1:(length(n))){
    convolution[i]<-sum(n[1:i]*n[i:1])
  }
  for(i in 1:(length(n))){
    convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)
       ]*n[length(n)+1-(i:1)])
  }
  return(convolution)
}

d24<-ConvolveSelf(ans2)
d48<-ConvolveSelf(d24)
d96<-ConvolveSelf(d48)
plot(dist1,d96[241:2240])
```

Question 15

Let X follow an exponential distribution with mean θ. Approximate this with an arithmetic distribution ($h = 1$) using:
(a) The method of rounding.
(b) The method of local moment matching, matching 2 moments on each interval.

Question 16

The loss on a given policy is modelled as following an exponential distribution with mean 2,000. The number of losses follows a negative binomial distribution with parameters $r = 4$ and $\beta = 2.1$.
(a) Calculate the distribution of the aggregate loss.
(b) What effect would a deductible of \$500 have on this distribution?

Question 17

The loss on a given policy is modelled as following a gamma distribution with $\alpha = 3.4$ and $\theta = 2000$. The number of losses an insurance company insures follows a Poisson distribution with $\lambda = 100$. The company has taken out stop-loss insurance with a deductible of \$1,000,000. This insurance is priced at the expected payment on the policy plus one standard deviation.

(a) How much does the company pay for this reinsurance?

(b) How much should it pay if it introduces a \$1,000 deductible on these policies?

Question 18

In a group life insurance policy, a life insurance company insures 10,000 individuals at a given company. It classifies these workers in the following classes:

Type of worker	number	average annual probability of dying	average death benefit
Manual Laborer	4,622	0.01	\$100,000
Administrator	3,540	0.002	\$90,000
Manager	802	0.01	\$200,000
Senior Manager	36	0.02	\$1,000,000

What is the probability that the aggregate benefit paid out in a year exceeds \$10,000,000?

Question 19

Using the same data as in Question 18, estimate the probability by modelling the distribution of the aggregate risk as:
(a) a normal distribution
(b) a gamma distribution
(c) a log-normal distribution

Question 20

Using the same data as in Question 18, estimate the probability using a compound Poisson approximation, setting the Poisson mean to:
(a) equal the Bernoulli probability
(b) match the probability of no loss

Question 21

An insurance company has the following portfolio of car insurance policies:

Type of driver	Number	Probability claim	mean of claim	standard deviation
Safe drivers	800	0.02	\$3,000	\$1,500
Average drivers	2100	0.05	\$4,000	\$1,600
Dangerous drivers	500	0.12	\$5,000	\$1,500

(a) Using a gamma approximation for the aggregate losses on this portfolio, calculate the cost of reinsuring losses above \$800,000, if the loading on the reinsurance premium is one standard deviation above the expected claim payment on the reinsurance policy.
(b) How much does the premium change if we use a normal approximation?

Question 22

An insurance company assumes that for smokers, the claim probability is 0.02, while for non-smokers, it is 0.01. A group of mutually independant lives has coverage of 1000 per life. The company assumes that 20% of lives are smokers. Based on this assumption, the premium is set equal to 110% of expected claims. If 30% of the lives are smokers, the probability that claims will exceed the premium is less than 0.2. Using a normal approximation, determine the minimum number of lives in the group.

Question 23

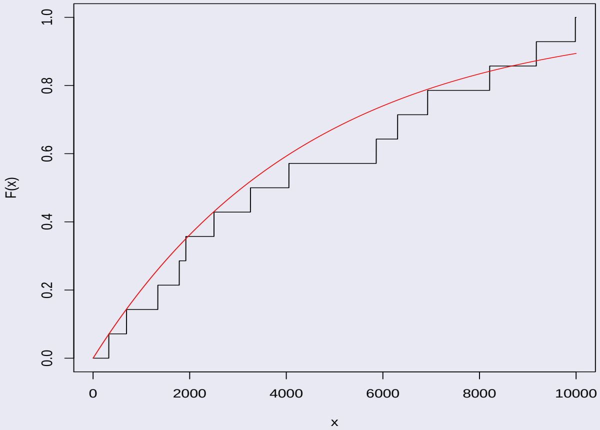
An insurance company is modeling claim severity. It collects the following data points:

325	692	1340	1784	1920	2503	3238	4054	5862
6304	6926	8210	9176	9984				

By graphically comparing distribution functions, assess the appropriateness of a Pareto distribution for modeling this data.

16.3 Graphical Comparison of Density and Distribution Functions

Answer to Question 23



16.3 Graphical Comparison of Density and Distribution Functions

Question 24

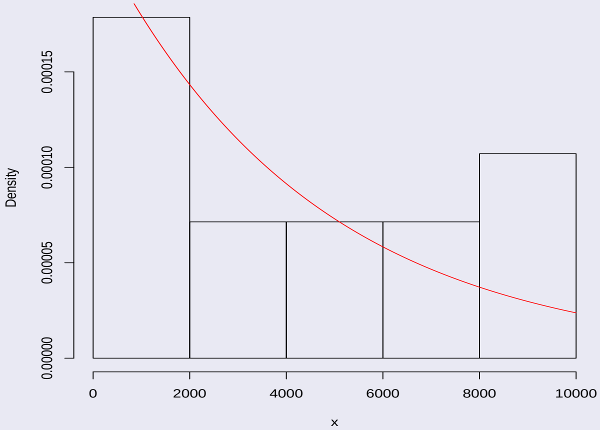
For the data from Question 23:

325 692 1340 1784 1920 2503 3238 4054 5862
6304 6926 8210 9176 9984

Graphically compare density functions to assess the appropriateness of a Pareto distribution for modeling this data.

16.3 Graphical Comparison of Density and Distribution Functions

Answer to Question 24



16.3 Graphical Comparison of Density and Distribution Functions

Question 25

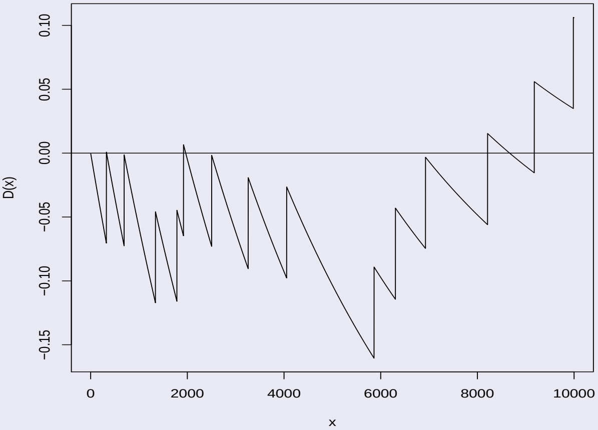
For the data from Question 23:

325	692	1340	1784	1920	2503	3238	4054	5862
6304	6926	8210	9176	9984				

By Graphing the differnce $D(x) = F^*(x) - F_n(x)$, assess the appropriateness of a Pareto distribution for modeling this data.

16.3 Graphical Comparison of Density and Distribution Functions

Answer to Question 25



16.3 Graphical Comparison of Density and Distribution Functions

Question 26

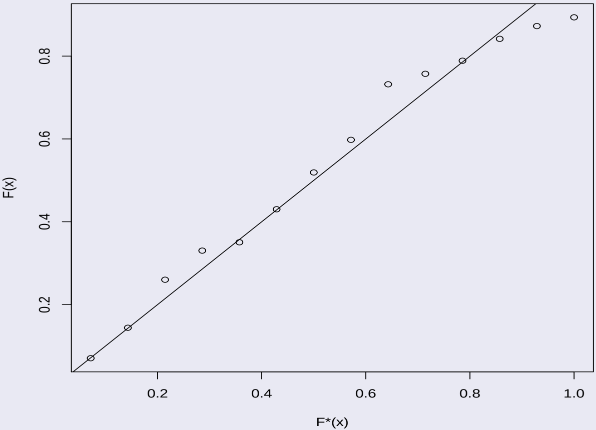
For the data from Question 23:

325	692	1340	1784	1920	2503	3238	4054	5862
6304	6926	8210	9176	9984				

Use a *p*-*p* plot to assess the appropriateness of a Pareto distribution for modeling this data.

16.3 Graphical Comparison of Density and Distribution Functions

Answer to Question 26



16.3 Graphical Comparison of Density and Distribution Functions

Question 27

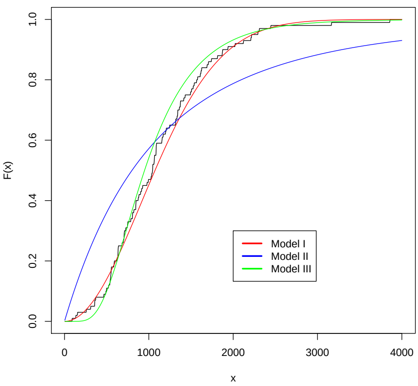
An insurance company is modelling a data set. It is considering 3 models, each with 1 parameter to be estimated. On the following slides are various diagnostic plots of the fit of each model.

Determine which model they should use for the data in the following situations. Justify your answers.

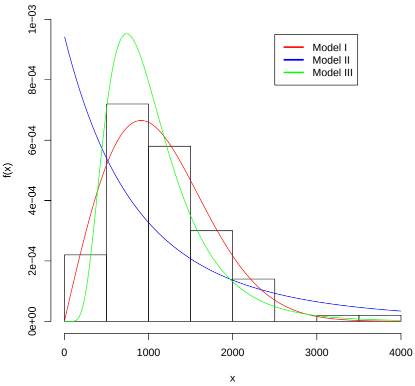
(a) Which model should they choose if accurately estimating (right-hand) tail probabilities is most important?

(b) The company is considering imposing a deductible, and therefore wants to model the distribution very accurately on small values of x.

Models



F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
1000
2000
3000
4000
x
Model I
Model II
Model III



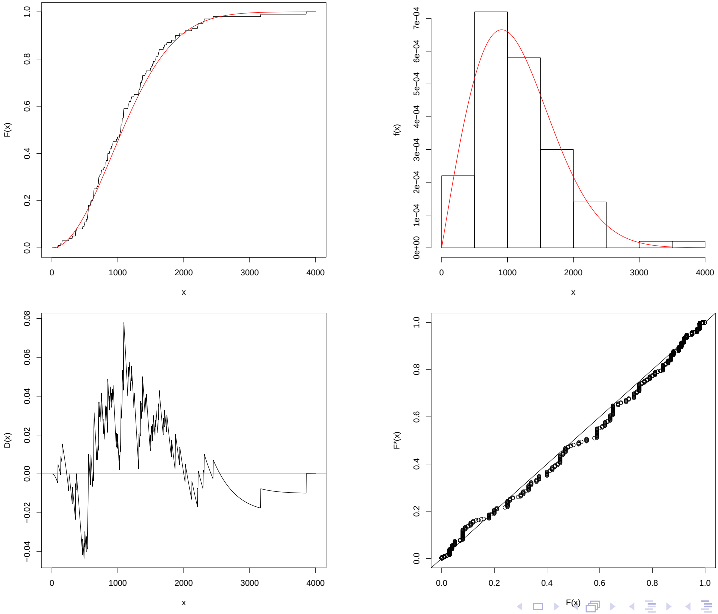
f(x)
0e+00
2e−04
4e−04
6e−04
8e−04
1e−03
0
1000
2000
3000
4000
x
Model I
Model II
Model III

Question 28

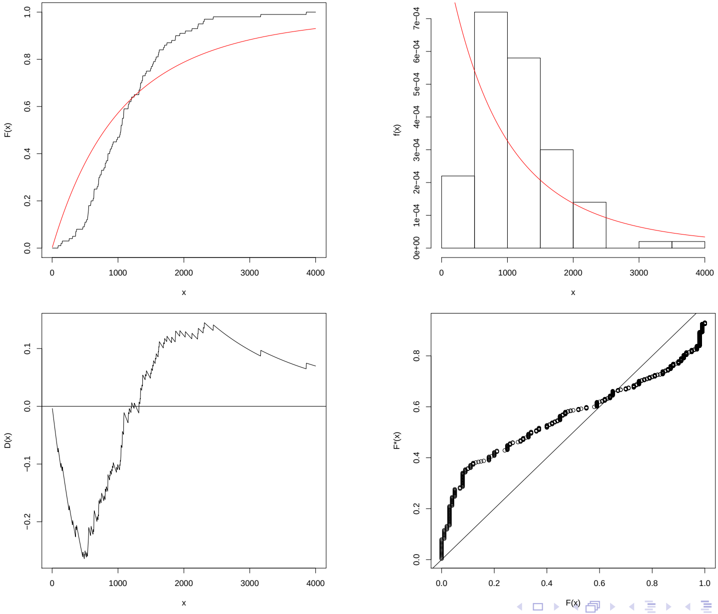
For each of the models on the following three slides, determine which of the statements below best describes the fit between the model and the data:

i. The model distribution assigns too much probability to high values and too little probability to low values.
ii. The model distribution assigns too much probability to low values and too little probability to high values.
iii. The model distribution assigns too much probability to tail values and too little probability to central values.
iv. The model distribution assigns too much probability to central values and too little probability to tail values.

Model I

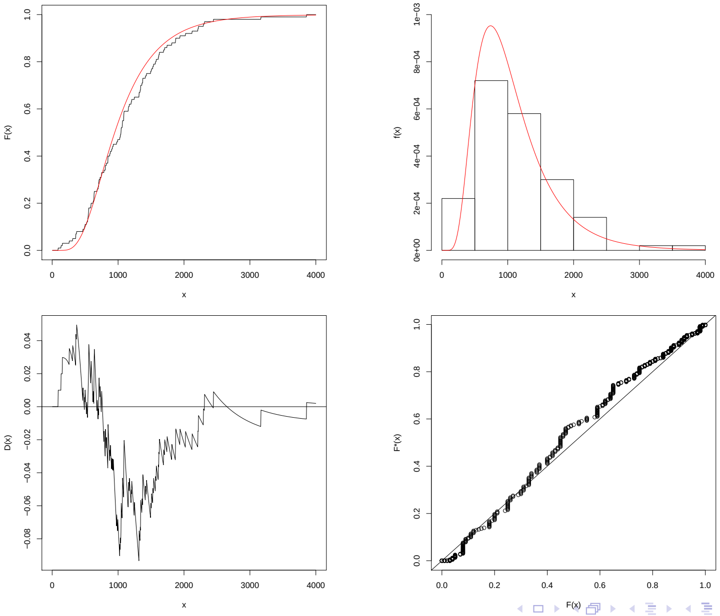


Model II



F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
1000
2000
3000
4000
x
f(x)
0e+00
1e−04
2e−04
3e−04
4e−04
5e−04
6e−04
7e−04
0
1000
2000
3000
4000
x
D(x)
−0.2
−0.1
0.0
0.1
0
1000
2000
3000
4000
x
F*(x)
0.0
0.2
0.4
0.6
0.8
0.0
0.2
0.4
0.6
0.8
1.0
F(x)

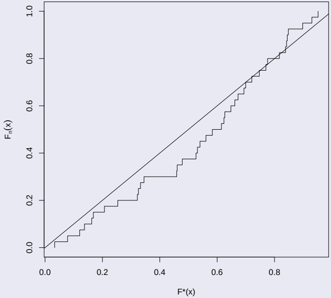
Model III



16.3 Graphical Comparison of Density and Distribution Functions

Question 29

An insurance company wants to know whether an exponential distribution is a good fit for a sample of 40 claim severities. It estimates $\theta = 5.609949$, and draws the following p-p plot:

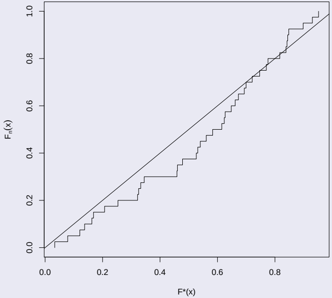


How many of the samples they collected were more than 10?

16.3 Graphical Comparison of Density and Distribution Functions

Question 30

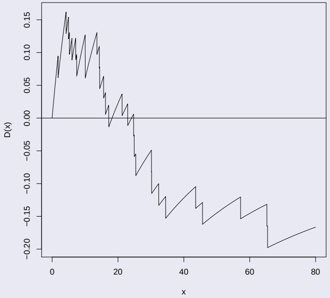
An insurance company wants to know whether an exponential distribution is a good fit for a sample of 40 claim severities. It estimates $\theta = 5.609949$, and draws the following p-p plot:



How many of the samples they collected were less than 3?

Question 31

An insurance company wants to know whether a Pareto distribution with $\theta = 15$ is a good fit for a sample of 30 claim severities. It estimates $\alpha = 0.8725098$ and draws the following plot of $D(x)$:



How many of the samples they collected were less than 10?

16.4 Hypothesis Tests

Hypothesis Tests

We test the following hypotheses:

H_0: The data came from a population with the given model.

H_1: The data did not come from a population with the given model.

16.4 Hypothesis Tests

Kolmogorov-Smirnov test

$$D = \max_{t \leqslant x \leqslant u} |F_n(x) - F(x)|$$

Anderson-Darling test

$$A^2 = n \int_t^u \frac{(F_n(x) - F(x))^2}{F(x)(1 - F(x))} f(x)\, dx$$

Chi-square Goodness-of-fit test

- Divide the range into separate regions, $t = c_0 < c_1 < \cdots < c_n = u$.
- Let O_i be the number of samples in the interval $[c_{i-1}, c_i)$.
- Let E_i be the expected number of sample in the interval $[c_{i-1}, c_i)$.

$$X^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

Question 32

For the data from Question 23:

325	692	1340	1784	1920	2503	3238	4054	5862
6304	6926	8210	9176	9984				

Test the goodness of fit of the model using:
(a) The Kolmogorov-Smirnov test.
(b) The Anderson-Darling test.

Answer to Question 32

(b)

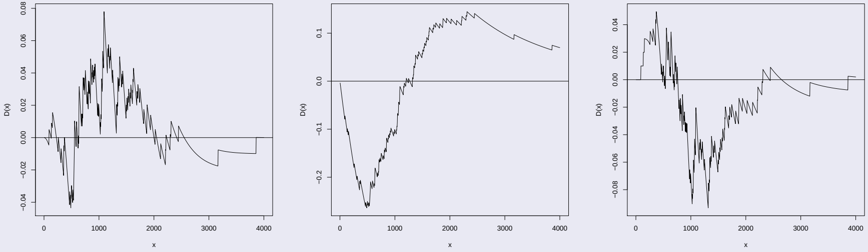
$$A^2 = -nF^*(u) + \sum_{j=0}^{k}(1 - F_n(y_j))^2(\log(1 - F^*(y_j)) - \log(1 - F^*(y_{j+1}))$$

$$+ n\sum_{j=1}^{k} F_n(y_j)^2(\log(F^*(y_{j+1})) - \log(F^*(y_{j+1})))$$

x	$F_n(x)$	$F^*(x)$	term	x	$F_n(x)$	$F^*(x)$	term
325	0.0714	0.0704	0.0748	4054	0.5714	0.5978	0.1407
692	0.1429	0.1440	0.1190	5862	0.6429	0.7320	0.0267
1340	0.2143	0.2600	0.0726	6304	0.7143	0.7573	0.0323
1784	0.2857	0.3303	0.0204	6926	0.7857	0.7889	0.0532
1920	0.3571	0.3504	0.0803	8210	0.8571	0.8417	0.0309
2503	0.4286	0.4302	0.0876	9176	0.9286	0.8726	0.0215
3238	0.5000	0.5169	0.0822	9984	1.0000	0.8937	0.1124

Question 33

Recall Question 27, where a company was deciding between three models. The $D(x)$ plots are below:



If the company uses the Kolmogorov-Smirnov statistic to decide the best model, which will it choose?

Question 34

An insurance company records the following claim data:

Claim Amount	Frequency
0–5,000	742
5,000–10,000	1304
10,000–15,000	1022
15,000–20,000	830
20,000–25,000	211
More than 25,000	143

Use a Chi-square test to determine whether Claim size follows an exponential distribution.

Likelihood Ratio test

The Likelihood ratio test compares two nested models — $\mathcal{M}_0$ and $\mathcal{M}_1$.

Hypotheses

H_0: The simpler model describes the data as well as the more complicated model.

H_1: The more complicated model describes the data better than the simpler model.

We compute the parameters from both models by maximum likelihood. The test statistic is.

$$2(l_{\mathcal{M}_0}(x;\theta_0) - l_{\mathcal{M}_1}(x;\theta_1))$$

Under H_0, for large n, this follows a Chi-square distribution with degrees of freedom equal to the difference in number of parameters.

Question 35

An insurance company observes the following sample of claim data:

382	596	920	1241	1358	1822	2010	2417	2773
3002	3631	4120	4692	5123				

Use a likelihood ratio test to determine whether an exponential or a Weibull distribution fits this data better.

Study note: Information Criteria

Basic Idea

- For natural measures of fit (log-likelihood, KS test statistic, AD test statistic, etc.) more complicated models produce better fit.
- This is (at least partly) because they are fitting noise in the data.
- We can compensate for this by adding a penalty term to penalise model complexity.

Two Common Approaches

- Akaike Information Criterion (AIC): $l(\theta; x) - 2p$
- Schwarz Bayesian Criterion(SBC)/Bayesian Information Criterion (BIC): $l(\theta; x) - \frac{p}{2}\log(n)$

where p is the number of estimated parameters, and n is the sample size.

Question 36

Recall Question 35, where we had a sample

382	596	920	1241	1358	1822	2010	2417	2773
3002	3631	4120	4692	5123				

for which the Weibull distribution has a log-likelihood of -120.7921. Use AIC and BIC to determine whether an inverse exponential distribution is a better fit for the data.

16.5 Selecting a Model

Comments on Model Selection

- Try to pick a model with as few parameters as possible. (Parsimony)
- Choice of model depends on the aspects that are important. Even if a formal test is used, the choice of which test depends on the aspects that are important.
- Aim is generalisability. The model should apply to future data. (Models which fit the given data well, but not new data are said to overfit.)
- Trying large numbers of models will lead to one which fits well just by chance.
- Experience is a valuable factor in deciding on a model.
- Sometimes knowledge of the underlying process may lead to a particular model (e.g. binomial).

2.2 Automobile insurance

Types of Automobile-insurance coverage

- Liability insurance
- Medical Benefits
- Uninsured and underinsured motorist coverage
- Collision insurance
- Other than collision (OTC) insurance

Notes

- First two coverages often legally required.
- Other coverages may be required in order to use car as security for a loan.
- Policy usually covers policyholder and immediate family on listed vehicles. May also cover invited drivers.
- Covers listed vehicles and usually also attached trailers, etc.
- Other policies apply to commercial vehicles.

2.2 Automobile insurance

Tort system

In this system, fault for an accident is legally determined through a court case, or settlement. At-fault party is legally responsible for all costs.

No-fault system

Injured party is covered through their own insurance, with no need to determine fault for the accident. Many different details for how this system can work — e.g. threshold no-fault or government monopoly. Even in this system, determination of fault may be performed to determine future premiums.

Question 37

What are the advantages and disadvantages of each system?

Liability insurance

- Covers costs of policyholder's damage to third parties.
- Can cover legal costs of policyholder.
- Policy limits usually apply to damage payments, not legal costs.
- Insurer may however stop paying legal costs once its damage payments have already exceeded the policy limit.
- This insurance is compulsory almost everywhere, with minimum legal limits. It may be advisable to buy increased policy limits if these are low.
- Premiums vary with policy limits, location, use of automobile, driving record of policyholder, age, sex and marital status of the policyholder.

Medical benefits

- Sometimes called medical payments, personal injury protection or accident benefits.
- Covers costs (e.g. medical costs, income replacement, survivors benefits, rehabilitation costs, home care costs) arising from injury to policyholder (unless another party is liable).
- Usually compulsory
- May not apply to commercial vehicles where these benefits are covered by workers compensation.
- In tort juristictions, liability insurance is more important (and accounts for more of the costs). In no-fault juristictions, medical benefit insurance is more important, and accounts for more costs.

Uninsured and underinsured motorist coverage

- Covers costs to policyholder if injured by:
  - an unidentified driver
  - an uninsured driver
  - an underinsured driver (if liable driver's coverage is lower than policyholder's)

2.2 Automobile insurance

Collision insurance and OTC insurance

- Cover damage to policyholder's vehicle.
- loss is defined as lesser of cash value of damaged property or cost to repair/replace. May include special provisions for cases where cash value (with depreciation) exceeds outstanding balance on a loan secured by the property.
- Usually include a deductible.
- In a tort juristiction, the insurer who pays the benefits can sue the at-fault driver. If the suit is successful, the insurer will reimburse the deductible to the policyholder, and use the rest to cover the payment. This is known as subrogation.
- Subrogation reduces the cost of collision insurance, and increases the cost of liability insurance.
- If the insurer decides not to sue, the policyholder can sue for their deductible, and any costs exceeding the policy limit.

2.2 Automobile insurance

Collision insurance and OTC insurance (cont.)

- If vehicle is "written-off", the insurer has the right of salvage — any scrap value of vehicle belongs to insurer. If scrap value exceeds amount paid, the insurer must increase its payment accordingly.
- Premiums for collision and OTC insurance depend on:
  - Type of car (based on value of car and cost of repairs).
  - Use of car
  - Location.
  - Driver's history.
  - Where allowed by law: age, gender, marital status.
- OTC covers fire, weather, vandalism, stone chips, theft, etc.
- Usually excludes: war, terrorism, wear & tear, road damage to tyres, radioactive contamination, and collision.
- Comprehensive includes any cause not specifically excluded. Under specified perils, only a given list of causes are reimbursed.
- OTC premiums usually only vary by vehicle type and location.

Four Coverages in Homeowner's Insurance

- Coverage A — Damage to home
- Coverage B — Damage to garage or other structures
- Coverage C — Personal property in home
- Coverage D — Living expenses and loss of rental income
- Section II — Liability

Doctrine of Proximate Cause

- Coverage might be comprehensive or specified perils.
- Proximate cause is a legal definition of when one event can be considered to have caused another. The proximate cause must be directly linked to the damage by a chain of events without other independent causes.
- For a claim to be payable, an insured peril must be a proximate cause of an insured loss.

2.3 Homeowner's insurance

Coverage A

- Includes a deductible (may decrease to zero for larger losses).
- Subrogation can apply here if a third party is liable for the damage.
- Policy limits less than 80% of the value of the house (at time of loss) result in coinsurance for smaller claims.
- Increases in house prices could result in a homeowner falling below the 80% cut-off by accident.
- Many insurers offer an option for the premium to increase annually in line with a specified index, and to waive the 80% requirement.

Question 38

A homeowner's house is valued at \$350,000. However, the home is insured only to a value of \$260,000. The insurer requires 80% coverage for full insurance. The home sustains \$70,000 of water damage due to a burst pipe. How much does the insurer reimburse? (There is no deductible for losses above \$2,000.)

2.3 Homeowner's insurance

Coverage B — Garage

- Usually up to 10% value of house.
- More coverage can be purchased for extra premium.

Coverage C — Personal Property

- Limit usually 40–50% of house value
- Often inside limits on each type of item.
- For full insurance on jewellry, silverware or art, policyholder can provide a schedule showing appraised values of these items. If lost or stolen, this appraised value is paid by insurer (not the current market value).
- Extends to borrowed property in policyholder's possession.
- Also applies to personal property lost or damaged outside the home.

2.3 Homeowner's insurance

Coverage D — Accommodation and Loss of Rental Income

- Covers fair rental value for alternative accommodation while repairs are made to home.
- Also covers lost rental income from any part of the home that lost while damage is repaired.

Section II — Liability

- Liability could arise if a third party is injured or property is damaged while on the property.
- Only applies in case of negligence by homeowner.
- Most claims settled out of court.
- Insurance will pay policyholder's defence costs unless liability payments already made exceed policy limits.
- Limited medical coverage for injured third-parties on a no-fault basis.

Comments on Homeowner's Insurance

- Premiums depend on location, construction and value.
- In high-risk areas for floods and earthquakes, these perils are often excluded.
- Coverage for these excluded perils can be purchased for an extra premium.
- Construction may affect the risk of various perils.
- Discounts may be offered for security systems or sprinkler systems, etc.

2.4 Tennants package

Tennant insurance

- Contains provisions of homeowners insurance relevant to tennants
- Includes personal possessions.
- Generally lower liability provisions for apartments because majority of liability arises from incidents on surrounding ground, covered by landlord's insurance.
- Different policies also available for condominium owners.

2.5 Workers' compensation

Workers' Compensation

- Early type of no-fault insurance.
- Prior to 1895, getting compensation was difficult for employees.
- Objectives of Worker's Compensation:
  - Broad coverage of occupational illness and injury.
  - Protection against loss of income
  - Provide medical care and rehabilitation expenses
  - Encourage employers to provide safer workplaces.
  - Provide efficient and effective delivery of benefits.
- Workers' Compensation Board controlled by province.
- U.S. employers can choose private, self or state insurance.
- Employee must work in a covered occupation, and experience an accident or disease resulting from employment, while employed.
- Diseases develop slowly, generally more expensive, often subject to disagreements about extent to which they are caused by work.

Typical Benefits

- Unlimited medical care benefits
- Disability income benefits, waiting period 3–7 days, percentage of salary, depending on degree of disability. Degree of disability is usually classified as:
    - Temporary but total
    - Permanent and total
    - Temporary and partial
    - or Permanent but partial
- Death benefits
- Rehabilitation benefits and services.
- Premiums depend on salaries of employees, industry class, etc.

2.6 Fire insurance

Fire insurance

- Included in homeowner insurance and tennants insurance.
- Policies provide protection for commercial properties.
- Originally only fire was an insured peril, but many more perils now covered, even some comprehensive policies.
- Covers both direct and indirect loss.
- Standard Fire Policy covers direct loss from fire and lightning.
- At least one additional form must be added. Common forms:
  - Include personal coverage
  - Include commercial coverage
  - Increase covered perils (e.g. vandalism, malicious mischief, etc.)
  - Increase covered losses (e.g. living expenses, rental income, leasehold interests, demolition, business interruption losses)
- Allied lines are additional coverage sold in separate policies, e.g. earthquake, rain, sprinkler leakage, water damage, crop hail.
- Larger corporations may design own forms to meet specific needs.

2.7 Marine insurance

Ocean Marine Insurance

- Covers oceangoing ships and cargo. Covers shipowner's liability.
- Basic policy only covers cargo while loaded onto ship.
- Some policies provide warehouse-to-warehouse coverage.

Inland marine insurance

- A modification of marine insurance for the trucking industry.
- Covers transportation of goods by railway, motor vehicle, ship or barge on inland waterways (e.g. canals and rivers) or coastal trade, air, mail, armoured car or messenger.
- Also covers infrastructure for transportation — bridges, tunnels, wharves, docks, communication equipment, moveable property.
- May have additional coverage for construction equipment, personal jewellry and furs, agricultural equipment, and animals.

2.8 Liability insurance

Types of Liability Insurance

- Included by default in auto and homeowner's insurance.
- Product liability insurance
- Errors and omissions Insurance
- Medical malpractice insurance
- Professional liability insurance

Features of Liability Insurance

- Low frequency high value claims.
- Claims often reported years after event.
- Claims often take many more years to settle after reporting.
- Claims-made policy form covers only claims after a specified date reported during policy period.
- Tail coverage sold to cover claims reported after the period.
- High litigation cost. Sometimes policy limit applies to legal costs.

2.9 Limits to Coverage— Deductibles

Reasons for Using Deductibles

- Small claims involve disproportionate administrative costs.
- Premium savings.
- Moral hazard.
- Better expected utility.

Problems with Deductibles

- Public relations.
- Marketing difficulties.
- Insureds may inflate claims to recover deductible.

Types of Deductible

- Fixed dollar deductibles
- Fixed %age deductibles
- Disappearing deductibles
- Franchise deductibles
- Fixed dollar deductible per year
- Elimination period

2.9 Limits to Coverage— Policy Limits

Reasons for Policy Limits

- Clarifies insurer's oblications
- Reduces risk to insurer, allowing lower premiums.
- Allows policyholder to choose appropriate coverage.

Notes on policy limits

- Policy may have different policy limits for different parts of claim.
- May also have inner limits.
- Policy limits apply to damage payments only — they do not include administrative and legal costs.

Question 39

An insurer sells 100 identical policies. Each policy has a 50% probability of incurring a loss. If a loss is incurred, the loss follows a Pareto distribution with $\alpha = 2$ and $\theta = 10000$.
(a) What is the probability that at least one policy incurs a single loss exceeding \$1,000,000?
(b) If the insurer has a policy limit of \$100,000 on the policy, what is the probability that the aggregate loss exceeds \$1,000,000?

4.2 How outstanding claim payments arise

Typical Steps in a Claim Payment

1. Claim Event
2. (a) Claim reported to agent. (b) Claim recorded by insurer.
3. (a) Initial payments and settlement offer. (b) Settlement rejected.
4. Court case.

Notes

- Common for lengthy delays to occur between steps.
- Claims could remain open for 10–20 years.
- Time to settlement can vary a lot with line of insurance.
- Largest claims often settle last.

Sources of Uncertainty

- Eventual cost of claim payments for known claims.
- Claims incurred but not yet reported.

4.3 Definition of terms

Claim File

- Established as soon as field adjuster is aware of pending claim.
- Field adjuster estimates expected ultimate claim payment, and updates the estimate regularly.
- Aggregate of individual claim file estimates called case reserves.

Gross IBNR

Additional reserves above case reserves, also called bulk reserve, including provision for:

- Adjustments of case reserves
- Claim files which are closed but may reopen.
- Claims incurred but not reported (IBNR).
- Claims reported but not recorded (RBNR).

4.3 Definition of terms

Paid Loss Development

- Change in cumulative payments made between valuation dates called paid age-to-age loss development.
- Relative change called paid loss development factor. Can be less than 1 because of salvage and subrogation.
- Difference between cumulative payments made and ultimate payment amount called age-to-ultimate loss development.

Incurred Loss Development

- Incurred age-to-age loss development: change in estimated costs.
- Incurred loss-development factor is relative change.
- Incurred loss-development factors can be less than 1 if estimates were conservative or because of salvage and subrogation. Incurred loss-development factors greater than 1 indicate inadequate claim file estimates.

4.3 Definition of terms

Salvage and Subrogation

Recall that after paying a claim, insurance company acquires rights of salvage and subrogation (provided the money returned does not exceed the claim amount). These can reduce the incurred losses.

Loss adjustment expenses

- Expenses involved with settling claims — e.g. legal costs.
- Allocated loss adjustment expenses (ALAE) relate to specific claims. These become part of total claim cost.
- Unallocated loss adjustment expenses (ALAE) (e.g. rent) are allocated among calculated reserves following a formula.
- Classification can vary between insurers.

Fast Track Reserves

High frequency, low severity lines of insurance use fast track average reserves based on recent experience and trends for new claim files.

4.4 Professional considerations

Setting loss reserves requires detailed knowledge

- Company's business. For example any changes in
  - Portfolio composition.
  - Claim administration.
  - Management.
  - Reinsurance.
- External factors. For example
  - Inflation.
  - Legal rulings.
- Format and definitions of data used by company. For example, how are claims separated into claim files? Actuary should separate data into homogeneous categories. This may involve splitting separate parts of individual claims.
- The actuary must review accuracy of data and compare multiple methods for estimating reserves. Where methods give conflicting answers, actuary must explain differences.

Checking for Inconsistencies.

- It is important to check for consistency of patterns accross the data.
- Where inconsistencies arise in the data, the actuary must identify the source of the inconsistency.
- Reserving actuary must document the findings and any ensuing adjustments or subjective changes to the calculations.

4.6 Loss reserving methods

Expected Loss Ratio Method

1. Calculate the expected ultimate loss ratio (ultimate claim payments divided by total earned premiums).
2. Multiply be earned premiums for period.
3. Subtract payments made to date.

Problems with this approach

- Danger of manipulation of expected loss ratio.
- Expected loss ratio will not apply after changes of premium.
- Expected loss ratio might change if portfolio changes.

Why is the method used?

- For new lines of business without past data, it may be the only available method.
- Can be used as a backup check for other methods.

Question 40

An insurance company has three types of claims with different expected loss ratios as shown in the following table:

Claim Type	Policy Year	Earned Premiums	Expected Loss Ratio	Losses paid to date
Collision	2014	$200,000	0.79	$130,000
	2015	$250,000	0.79	$110,000
	2016	$270,000	0.77	$60,000
Comprehensive	2014	$50,000	0.74	$36,600
	2015	$60,000	0.72	$44,300
	2016	$65,000	0.75	$41,400
Bodily Injury	2014	$300,000	0.73	$86,000
	2015	$500,000	0.73	$85,000
	2016	$600,000	0.72	$12,000

Use the expected loss ratio method to estimate the loss reserves needed.

Question 41

The following table shows the paid losses on claims from one line of business of an insurance company over the past 6 years.

	Development year					
Accident year	0	1	2	3	4	5
2011	5,826	659	2,910	845	1,349	120
2012	7,327	2,896	1,540	963	−348	
2013	8,302	1,719	1,380	2,031		
2014	8,849	1,701	673			
2015	9,950	320				
2016	11,290					

Assume that all payments on claims arising from accidents in 2011 have now been settled. Estimate the future payments arising each year from open claims arising from accidents in each calendar year. (That is, fill in the empty cells in the table.)

Answer to Question 41

We first construct the cumulative loss payments in the following table:

Accident	Development year					
year	0	1	2	3	4	5
2011	5,826	6,485	9,395	10,240	11,589	11,709
2012	7,327	10,223	11,763	12,726	12,378	
2013	8,302	10,021	11,401	13,432		
2014	8,849	10,550	11,223			
2015	9,950	10,270				
2016	11,290					

Answer to Question 41

The corresponding loss development factors are

Accident year	Development year 1/0	2/1	3/2	4/3	5/4
2011	1.113	1.449	1.090	1.132	1.010
2012	1.395	1.151	1.082	0.973	
2013	1.207	1.138	1.178		
2014	1.192	1.064			
2015	1.032				

Answer to Question 41

The corresponding loss development factors are

Accident year	Development year 1/0	2/1	3/2	4/3	5/4
2011	1.113	1.449	1.090	1.132	1.010
2012	1.395	1.151	1.082	0.973	
2013	1.207	1.138	1.178		
2014	1.192	1.064			
2015	1.032				

4.6 Loss reserving methods— Chain ladder or loss development triangle method

Approaches

- Average
- 5-year average
- Mean — weighted by claim payment amounts.

Pros and Cons of Incurred Loss Triangles

- Incurred loss estimates represent the company's best estimate of losses, including information not reflected in paid loss data.
- Paid loss data are objective, incurred loss data are subjective.
- Incurred loss estimates react immediately to changes.

Problems with chain ladder approach

- Too many parameters for the data.
- Unstable — changes in methodology or a few large claims can have excessive influence on estimated reserves.

Bornhuetter-Ferguson method

1. Calculate the expected ultimate claim payments (using expected ultimate loss ratio times earned premiums)
2. Calculate loss development factors using chain-ladder method
3. Work backwards from expected ultimate payments using loss development factors to get expected loss development.

Question 42

Recall Question 41, where the average loss development factors were

Year	1/0	2/1	3/2	4/3	5/4
Average	1.113114	1.200218	1.11665	1.052196	1.010355

Suppose the expected loss ratio is 0.72, and the earned premiums are

Accident Year	2012	2013	2014	2015	2016
Earned Prem.	180,000	205,000	210,000	220,000	270,000

Use the Bornhuetter Fergusson method to calculate the loss reserves needed for each accident year.

Question 43

An actuary is reviewing the following loss development triangles:

No. of closed claims

Acc. Year	Development Year 0	1	2	3	4	Ult.
2012	250	335	370	395	400	400
2013	280	385	400	450		460
2014	330	395	470			500
2015	320	460				540
2016	360					580

Total paid losses on closed claims (000's)

Acc. Year	Development Year 0	1	2	3	4
2012	723	2,087	2,263	2,822	4,783
2013	1,509	2,641	2,948	5,256	
2014	1,745	3,214	3,754		
2015	3,094	3,244			
2016	2,824				

Calculate tables of percentage of claims closed and cumulative average losses.

Answer to Question 43

percentage of claims closed

Acc. Year	Development Year 0	1	2	3	4
2012	62.5	83.8	92.5	98.8	100
2013	60.9	83.7	87.0	97.8	
2014	66.0	79.0	94.0		
2015	59.3	85.2			
2016	62.1				

cumulative average payment per claim

Acc. Year	Development Year 0	1	2	3	4
2012	2,892	6,230	6,116	7,144	11,958
2013	5,389	6,860	7,370	11,680	
2014	5,288	8,137	7,987		
2015	9,669	7,052			
2016	7,844				

Question 44

For the loss development triangles from Question 43:

Percentage of closed claims

Acc. Year	Development Year 0	1	2	3	4
2012	62.5	83.8	92.5	98.8	100
2013	60.9	83.7	87.0	97.8	
2014	66.0	79.0	94.0		
2015	59.3	85.2			
2016	62.1				

Total paid losses on closed claims (000's)

Acc. Year	Development Year 0	1	2	3	4
2012	723	2,087	2,263	2,822	4,783
2013	1,509	2,641	2,948	5,256	
2014	1,745	3,214	3,754		
2015	3,094	3,244			
2016	2,824				

Adjust the total paid losses to use the current disposal rate.

Answer to Question 44

Acc.	Development Year				
Year	0	1	2	3	4
2012	728	2,053	2,227	2,851	4,783
2013	1,480	2,595	2,728	5,256	
2014	1,855	2,980	3,754		
2015	2,954	3,244			
2016	2,824				

Notes on using separate frequency and severity

- Easier to see some patterns or trends.
- Can use this to adjust for different speeds of finalisation.
- Note that payments in this table represent payments made on closed files, and do not include partial payments on open files.
- When calculating the average, usually exclude claims on which no payment was made. If this is not possible, it is important to be consistent between different years.

4.7 Discounting loss reserves

Discounting Reserves

- Discounting future payments is a basic actuarial principal.
- However, many juristictions forbid discounting loss reserves.
- One argument in support of this is that it incorporates a contingency loading in the reserves.
- On the other hand, it may be better to incorporate an additional contingency loading.
- Timing of payments needs to be considered.
- Often a final small reserve should be retained for claims that take an extremely long time to settle. Estimating the timing of this can be difficult.
- Investment management is more complicated, and tax issues also arise.

Question 45

Recall that in Quesion 41 we calculated the following expected payments.

Accident year	Development year 0	1	2	3	4	5
2012						12506
2013					14133	14279
2014				12532	13186	13323
2015			12326	13764	14483	14632
2016		12567	15083	16843	17722	17905

Using a discount rate of 7%, calculate the expected reserves needed to cover these payments.

Answer to Question 45

The discounted values of all expected future payments are:

Accident year	Development year 0	1	2	3	4	5
2012						11,688
2013					13,100	12,370
2014				11,726	11,436	10,799
2015			11,272	11,777	11,486	10,846
2016		12,464	13,680	14,293	13,940	13,163

Example

- An insurance company offers group life insurance to all 372 employees of a company.
- The premium is set at $1,000 per year.
- The company notices that the average annual total claim over the past 7 years is $126,000 — Far lower than the total premiums charged.

The company contacts the insurers and asks for a reduction in premiums on the basis that premiums are much larger than the average claim.

(a) Is this request reasonable?

(b) What would be a fair reduction in premium?

17.3 Full Credibility

Definition

We assign full credibility to a policyholder's past history if we have sufficient data to use the policyholder's average claim for our premium estimate.

Criterion for Full Credibility

Let ξ be the (unknown) expected claim from a policyholder. We pick $r \geqslant 0$ and $0 < p < 1$. We assign full credibility to X if

$$P(|\overline{X} - \xi| < r\xi) > p$$

That is if with probability p, the relative error of $\overline{X}$ as an estimator for ξ is less than r.

Question 46

Recall our earlier example:

- An insurance company offers group life insurance to all 372 employees of a company.
- The premium is set at $1,000 per year.
- The average annual total claim over the past 7 years is $126,000.

Suppose that all policies have a death benefit of $98,000, and deaths of each employee are independent.
(a) Should the insurers assign full credibility to this experience? (Use $r = 0.05$ and $p = 0.95$.)
(b) How many years of past history are necessary to assign full credibility?

Question 47

Recall our earlier example:

- An insurance company offers group life insurance to all 372 employees of a company.
- The premium is set at $1,000 per year.
- The average annual total claim over the past 7 years is $1,260,000.

Suppose that all policies have a death benefit of $98,000, and deaths of each employee are independent.
If the standard for full credibility is measured in terms of number of claims, instead of number of years, what standard is needed in this case, and how does the standard vary with number of years.

Question 48

A car insurance company is reviewing claims from a particular brand of car. It finds that over the past 3 years:

- it has issued 41,876 annual policies for this type of car.
- The average annual aggregate claim per policy is \$962.14.
- The standard deviation of annual aggregate claim per policy is \$3,605.52

(a) Should it assign full credibility to the historical data from this type of car?
(b) How many policies would it need in order to assign full credibility?

Question 49

Recall our original example:

- Group life insurance for 372 employees of a company.
- The premium is set at $1,000 per year.
- The average annual total claim over the past 7 years is $126,000.

All policies have a death benefit of $98,000, and deaths of each employee are independent.
In Question 46, we determined that this was not sufficient to assign full credibility to the data, and that 1191.034 years of claims data would be needed for full credibility.

> How much credibility should we assign to this data, and what should the resulting premium be?

Question 50

For a particular insurance policy, the average claim is $230, and the average claim frequency is 1.2 claims per year. A policyholder has enrolled in the policy for 10 years, and has made a total of 19 claims for a total of $5,822. Calculate the new premium for this policyholder if the standards for full credibility are:

(a) 421 claims for claim frequency, 1,240 claims for severity.

(b) 1146 claims for claim frequency, 611 claims for severity.

(c) 400 years for aggregate losses

Problems

- No theoretical justification.
- Need to choose r and p arbitrarily.
- Doesn't take into account uncertainty in the book premium.

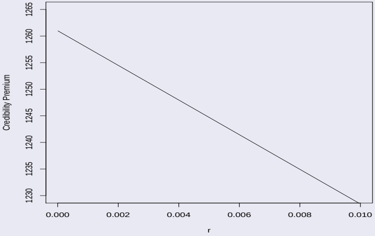
Question 51

An insurance company sells car insurance. The standard annual premium is \$1,261. A car manufacturer claims that a certain model of its cars is safer than other cars and should receive a lower premium. The insurance company has issued 3,722 policies for this model of car. The total aggregate claims on these polices were \$3,506,608. The variance of the annual aggregate claims on a policy is 8,240,268. Calculate the Credibility premium for different values of r and p.

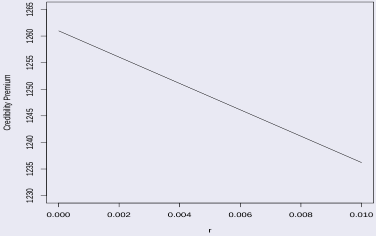
R-code for Question 51

```
#Limited Fluctuation Credibility Premium as r changes
p<-0.05
r<-(1:1000)/100000
Z<- 20.02297*r/qnorm(1-p/2)
Z<-pmin(Z,1)
plot(r,Z*3506608/3722+(1-Z)*1261,type='l')
pdf("LFCredibilityChangeRPp=0.05.pdf")
plot(r,Z*3506608/3722+(1-Z)*1261,type='l',ylab="
   Credibility Premium")
dev.off()
```

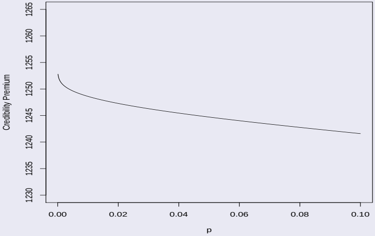
Answer to Question 51



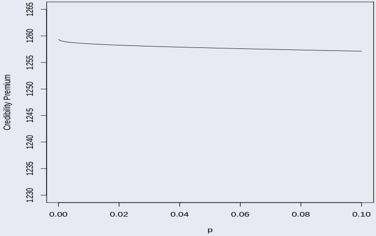
$p = 0.05$



$p = 0.01$

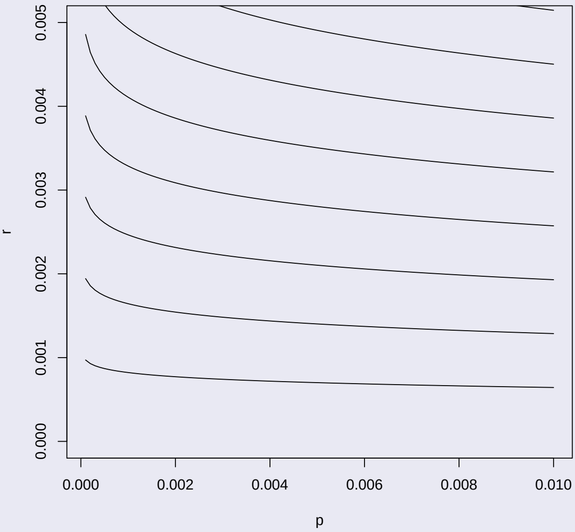


$r = 0.005$



$r = 0.001$

Answer to Question 51



Assumptions

- Each policyholder has a risk parameter Θ, which is a random variable, but is assumed constant for that particular policyholder.
- Individual values of Θ can never be observed.
- The distribution of this risk parameter Θ has density (or mass) function $\pi(\theta)$, which is known. (We will denote the distribution function $\Pi(\theta)$.)
- For a given value $\Theta = \theta$, the conditional density (or mass) of the loss distribution $f_{X|\Theta}(x|\theta)$ is known.

Conditional Distributions (revision)

$$f_{X|\Theta}(x|\theta) = \frac{f_{X,\Theta}(x,\theta)}{\int f_{X,\Theta}(y,\theta)\,dy}$$

$$f_{X|\Theta}(x|\theta) f_{\Theta}(\theta) = f_{\Theta|X}(\theta|x) f_X(x)$$

Conditional Expectation (revision)

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Theta))$$

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\Theta)) + \mathrm{Var}(\mathbb{E}(X|\Theta))$$

Question 52

An insurance company models drivers as falling into two categories: frequent and infrequent. 75% of drivers fall into the frequent category. The number of claims made per year by a driver follows a Poisson distribution with parameter 0.4 for frequent drivers and 0.1 for infrequent drivers.

(a) Calculate the expectation and variance of the number of claims in a year for a randomly chosen driver.

(b) Calculate the expectation and variance of the number of claims in a year for a randomly chosen driver who made no claims in the previous year.

Question 53

The aggregate health claims (in a year) of an individual follows an inverse gamma distribution with $\alpha = 3$ and θ varying between individuals. The distribution of θ is a Gamma distribution with parameters $\alpha = 3$ and $\theta = 100$.
(a) Calculate the expected total health claims for a random individual.
(b) If an individual's aggregate claims in two consecutive years are \$112 and \$240, calculate the expected aggregate claims in the third year.

Question 54

The number of claims made by an individual in a year follows a Poisson distribution with parameter Λ. Λ varies between individuals, and follows a Gamma distribution with $\alpha = 0.5$ and $\theta = 2$.
(a) Calculate the expected number of claims for a new policyholder.
(b) Calculate the expected number of claims for a policyholder who has made m claims in the previous n years.

Question 55

The number of claims made by an individual in a year follows a Poisson distribution with parameter Λ. Λ varies between individuals, and follows a Pareto distribution with $\alpha = 4$ and $\theta = 3$. [This has mean 1 and variance 2, like the Gamma distribution from Question 54.] Calculate the expected number of claims for a policyholder who has made m claims in the previous n years.

Answer to Question 55

	Pareto Prior					Gamma Prior			
	1	2	3	4		1	2	3	4
0	0.433	0.294	0.224	0.182	0	0.333	0.200	0.143	0.111
1	0.926	0.607	0.458	0.369	1	1.000	0.600	0.429	0.333
2	1.479	0.940	0.700	0.561	2	1.667	1.000	0.714	0.556
3	2.087	1.289	0.951	0.758	3	2.333	1.400	1.000	0.778
4	2.749	1.654	1.208	0.958	4	3.000	1.800	1.286	1.000
5	3.457	2.034	1.472	1.163	5	3.667	2.200	1.571	1.222
6	4.207	2.426	1.742	1.370	6	4.333	2.600	1.857	1.444
7	4.992	2.829	2.018	1.581	7	5.000	3.000	2.143	1.667
8	5.807	3.242	2.298	1.795	8	5.667	3.400	2.429	1.889
9	6.648	3.664	2.583	2.011	9	6.333	3.800	2.714	2.111

Problems with Bayesian Approach

- Difficult to Compute.
- Sensitive to exact model specification.
- Difficult to perform model selection for the unobserved risk parameter Θ.

Approach

- Credibility premium is a linear combination of book premium and personal history.

$$\alpha_0 + \sum_{i=1}^{n} \alpha_i X_i$$

- Coefficients are chosen to minimise Mean Squared Error (MSE)

$$\mathbb{E}\left(\mu(\Theta) - \left(\alpha_0 + \sum_{i=1}^{n} \alpha_i X_i\right)\right)^2$$

Question 56

Show that the solution which minimises the MSE satisfies:

$$\mathbb{E}(X_{n+1}) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \mathbb{E}(X_i)$$

$$\text{Cov}(X_i, X_{n+1}) = \sum_{j=1}^{n} \alpha_j \, \text{Cov}(X_i, X_j)$$

Question 57

Suppose the X_i all have the same mean, the variance of X_i is σ^2, and the covariance $\text{Cov}(X_i, X_j) = \rho$. Calculate the credibility estimate for X_{n+1}.

Question 58

Suppose we have observations $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, which are the aggregate annual claims for each of two cars driven by an individual. We assume:

$$\begin{aligned}
\mathbb{E}(X_i) &= \mu \\
\mathbb{E}(Y_i) &= \nu \\
\mathrm{Var}(X_i) &= \sigma^2 \\
\mathrm{Var}(Y_i) &= \tau^2 \\
\mathrm{Cov}(X_i, X_j) &= \rho \quad \text{for } i \neq j \\
\mathrm{Cov}(Y_i, Y_j) &= \zeta \quad \text{for } i \neq j \\
\mathrm{Cov}(X_i, Y_j) &= \xi
\end{aligned}$$

Calculate the credibility estimate for $X_{n+1} + Y_{m+1}$.

18.5 The Buhlmann Model

Assumptions

- $X_1, \ldots, X_n$ are i.i.d. conditional on Θ.

We then define:

$\mu(\theta) = \mathbb{E}(X|\Theta = \theta)$ $\quad\quad$ $\mu = \mathbb{E}(\mu(\Theta))$

$\nu(\theta) = \text{Var}(X|\Theta = \theta)$ $\quad\quad$ $\nu = \mathbb{E}(\nu(\Theta))$

$a = \text{Var}(\mu(\Theta))$

Solution

$$\mathbb{E}(X_i) = \mu \qquad \text{Var}(X_i) = \nu + a$$
$$\text{Cov}(X_i, X_j) = a$$

Recall from Question 57, that the solution to this is:

$$\hat{\mu} = \frac{\left(\frac{v}{a}\right)}{n + \left(\frac{v}{a}\right)}\mu + \frac{n}{n + \left(\frac{v}{a}\right)}\overline{X}$$

Question 59

An insurance company offers group health insurance to an employer. Over the past 5 years, the insurance company has provided 851 policies to employees. The aggregate claims from these policies are \$121,336. The usual premium for such a policy is \$326. The variance of hypothetical means is 23,804, and the expected process variance is 84,036. Calculate the credibility premium for employees of this employer.

Question 60

An insurance company offers car insurance. One policyholder has been insured for 10 years, and during that time, the policyholder's aggregate claims have been \$3,224. The book premium for this policyholder is \$990. The expected process variance is 732403 and the variance of hypothetical means is 28822. Calculate the credibility premium for this driver next year.

18.6 The Buhlmann-Straub Model

Assumptions

- Each observation X_i (expressed as loss per exposure) has a (known) exposure m_i. The conditional variance of X_i is $\frac{v(\theta)}{m_i}$.

$$\mathrm{Cov}(X_i, X_j) = a$$
$$\mathrm{Var}(X_i) = \frac{v}{m_i} + a$$

Solution

$$\alpha_0 = \frac{\left(\frac{v}{a}\right)}{m + \frac{v}{a}}\mu \qquad \alpha_i = \frac{m_i}{m + \frac{v}{a}}$$

$$\hat{\mu} = \frac{\left(\frac{v}{a}\right)}{m + \frac{v}{a}}\mu + \frac{m}{m + \frac{v}{a}}\overline{X}$$

where $\overline{X}$ is the weighted mean $\sum_{i=1}^{n} \frac{m_i}{m} X_i$.

Question 61

For a group life insurance policy, the number of lives insured and the total aggregate claims for each of the past 5 years are shown in the following table:

Year	1	2	3	4	5
Lives insured	123	286	302	234	297
Agg. claims	0	\$300,000	\$200,000	\$200,000	\$300,000

The book rate for this policy premium is \$1,243 per life insured. The variance of hypothetical means is 120,384 and the expected process variance is 81,243,100. Calculate the credibility premium per life insured for the next year of the policy.

Question 62

A policyholder holds a landlord's insurance on a rental property. This policy is in effect while the property is rented out. The company has the following experience from this policy:

Year	1	2	3	4	5	6
Months rented	3	11	8	12	6	9
Agg. claims	0	\$10,000	0	0	\$4,000	0

The standard premium is \$600 per year for this policy. The variance of hypothetical means is 832076, and the expected process variance is 34280533 (both for annual claims). Calculate the credibility premium for the following year using the Buhlmann-Straub model.

Question 63

Show that if the Bayes premium is a linear function of X_i, and the expectation and variance of X are defined, then the Bayes premium is equal to the credibility premium.

Question 64

Show that if the model distribution is from the linear exponential family, and the prior is the conjugate prior, with $\frac{\pi(\theta_1)}{r'(\theta_1)} = \frac{\pi(\theta_0)}{r'(\theta_0)}$, where θ_0 and θ_1 are the upper and lower bounds for θ, then the Bayes premium is a linear function in X.

Approach

- Estimate the distribution of Θ from the data.
- Use this estimate to calculate the credibility estimate of μ.

Two possibilities

Either: We do not have a good model for the conditional or prior distribution. We only need the variances, so we estimate them non-parametrically.

or: We have a parametric model, such as a Poisson distribution, which allows us to estimate the variance more efficiently (assuming the model is accurate).

Question 65

An insurance company has the following aggregate claims data on a new type of insurance policy:

No.	Year 1	Year 2	Year 3	Year 4	Year 5	Mean	Variance
1	336	0	528	0	0	172.80	60595.2
2	180	234	0	2,642	302	671.60	1225822.8
3	0	0	528	361	0	177.80	62760.2
4	443	729	1,165	0	840	635.40	192962.3
5	0	0	0	0	0	0.00	0.0
6	196	482	254	303	0	247.00	30505.0
7	927	0	884	741	604	633.60	140653.7
8	0	601	105	130	327	232.60	56385.3

(a) Estimate the expected process variance and the variance of hypothetical means.

(b) Calculate the credibility premiums for each policyholder next year.

Theorem

Let $X_1, \ldots, X_n$ all have mean μ, and let X_i have variance $\frac{\sigma^2}{m_i}$ where all m_i are known. Let $m = \sum_{i=1}^{n} m_i$.
We can obtain the following unbiassed estimators for μ and σ^2:

$$\hat{\mu} = \frac{\sum_{i=1}^{n} m_i X_i}{m}$$

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{n} m_i (X_i - \hat{\mu})^2}{n-1}$$

Question 66

An insurance company offers a group-life policy to 3 companies. These are the companies' exposures and aggregate claims (in millions) for the past 4 years:

Co.		Year 1	Year 2	Year 3	Year 4	Total
1	Exp	769	928	880	1,046	3,623
	Claims	1.3	1.5	0.8	1.7	5.3
2	Exp	1,430	1,207	949	1,322	4,908
	Claims	1.0	0.9	0.6	1.5	4.0
3	Exp	942	1,485	2,031	1,704	6,162
	Claims	1.1	1.4	1.9	2.0	6.4

Calculate the credibility premiums per life for each company in the fifth year.

Question 67

In a particular year, an insurance company observes the following claim frequencies:

No. of Claims	Frequency
0	3951
1	1406
2	740
3	97
4	13
5	3

Assuming the number of claims an individual makes follows a Poisson distribution, calculate the credibility estimate for number of claims for an individual who has made 6 claims in the past 3 years.

Question 68

Assume annual claims from one policyholder follow a Poisson distribution with mean Λ. The last 4 years of claims data are:

Claims	0	1	2	3	4	5	6	7	8	9
1 year	3951	1406	740	97	13	3	0	0	0	0
2 years	3628	2807	1023	461	104	13	4	0	1	0
3 years	2967	4032	2214	890	734	215	131	22	0	2
4 years	1460	2828	2204	985	747	358	194	43	8	0

Calculate the credibility estimate of Λ for an individual who made 2 claims in the last 3 years of coverage.

Question 69

Claim frequency in a year for an individual follows a Poisson with parameter Λt where Λ is the individual's risk factor and t is the individual's exposure in that year. An insurance company collects the following data:

Policy-holder	Year 1		Year 2		Year 3		Year 4	
	Exp	claims	Exp	claims	Exp	claims	Exp	claims
1	45	12	10	6	45	14	14	2
2	27	0	12	0	74	0	27	0
3	10	9	293	149	14	6	13	5
4	10	0	14	3	17	2	6	2

In year 5, policyholder 3 has 64 units of exposure. Calculate the credibility estimate for claim frequency for policyholder 3.

3.2 Objectives of ratemaking

Strict Requirements

- Cover expected losses and expenses.
- Make adequate provision for contingencies.
- Encourage loss control.
- Satisfy regulators.

Desirable Objectives

- Remain reasonably stable.
- Respond to changes.
- Be easy to understand.

3.3 Frequency and severity

Components needed to calculate rates

- Claim frequency distribution
- Loss distribution
- Interest rate
- Times of payments.

Sources of Uncertainty

- For life contingencies, loss is usually specified, so the main sources of uncertainty are claim frequency, interest rate and times of payment.
- For property or casualty insurance, claim frequency distribution and loss distribution are important sources of uncertainty.
- For lines of insurance where settlement can take a long time (e.g. liability insurance), time of payment and rate of interest can be a source of uncertainty.

3.4 Data for ratemaking— Three Ways to Record Data

Accident Year

- All payments are recorded under the year when the loss occurred.
- Data first becomes available on 31st December of that year.
- Data originally consists of paid loss amounts plus loss reserves.
- Data are updated as claims are settled.

Policy Year

- Payments recorded under the year when policy came into force.
- Data first available on 31st December of the following year.
- Data originally consists of paid loss amounts plus loss reserves.
- Has the advantage of being under the same policy basis.

Calendar Year

- Payments recorded under the year when the payment is made.
- Includes changes to loss reserves.
- Advantage is that it is finalised immediately.

Question 70

A home insurance policyholder pays $640 annual premium on 1st October 2015. What is the earned premium from this policy in
(a) 2015
(b) 2016

3.6 The exposure unit

Exposure Unit

- Measure of how exposed to loss the policy is.
- Premium calculated as $(\text{units of exposure}) \times (\text{rate per unit})$
- Examples include car-years, house value, payroll.

Good exposure units should

- Accurately measure exposure to loss
- Be easy to determine (at time of premium calculation)
- Be unable to be manipulated
- Be easy to administer
- Be easy for policyholder to understand.
- Automatically adjust with inflation.

3.7 The expected effective period

Question 71

An actuary is reveiwing claims data from accident year 2016 to calculate premiums for policy year 2018. She finds that the expected number of claims per unit of exposure is 0.003, and the expected claim value per claim in accident year 2016 was $26,000. Payments are subject to an annual inflation rate of 3%. What pure premium should she set for 2018?

3.8 Ingredients of ratemaking

Loss-Development Factors

- Work with incurred lossed (include estimated loss reserves)
- Adjust data to reduce impact of large losses.

Trend Factors

- Adjust expected premiums to future payment periods.
- Cover inflation, changes to court rulings, technical advances, etc.
- Usually estimated using least-squares (linear or non-linear) regression.
- Regression may be applied separately to frequency and severity, or to aggregate losses.
- Actuary may choose to assign more weight to more recent data.
- Should take into account external factors.
- Trends in premium can help estimate loss ratio method.

3.8 Ingredients of ratemaking

Expenses

- Usually divided between Loss Adjustment Expenses (LAE) and other expenses
- LAE are divided into allocated (ALAE) and unallocated (ULAE).
- Sometimes separate between expenses which are proportional to gross premiums and expenses proportional to exposure.

Loading for Profit and Contingencies

- Loading can be implicit or explicit (usually a percentage of gross premium).
- Implicit approach historically calculated by underestimating investment returns.
- Competition should prevent loading being too much.

Overall Rate Change

- Loss cost method: $\text{New average gross rate} = \frac{\text{New Average Loss Cost}}{1-\text{Expense Ratio}}$
- Loss ratio method: $\text{Rate Change} = \frac{\text{Expected Effective Loss Ratio}}{\text{Permissible Loss Ratio}} - 1$

Risk Classification Differential Changes

- Rate manual consists of rate for base cell, and for each variable, a vector of differentials — multiplicative factors.

Question 72

An insurer has three classes of risk - low, medium and high. Its experience from the previous year is shown in the table below.

Risk Class	Current differential	Earned premiums	Loss payments
Low	0.74	1,300	1,100
Medium	1	4,300	3,900
High	1.46	1,600	1,400

Calculate the new differentials for the coming year.

Question 73

An insurer has two risk variables — sex and risk level. Its current base rate is \$46.30 per unit of exposure. Its expense ratio is 20%. Its experience from the previous year is shown in the table below.

		Earned Premiums		Loss Payments	
		Male	Female	Male	Female
Differential		1	0.88	1	0.88
Low	0.74	900	1,100	1,050	850
Medium	1	4,700	4,400	4,100	3,900
High	1.46	1,900	1,400	1,200	1,100

Calculate the new rates for the coming year.

5.1 Individual risk rating plans

Experience rating

- Policyholder premium is credibility average of policyholder's experience and risk class rate.
- Usually set upper bound on impact of policyholder's experience.

Schedule rating

- Minor changes to premiums based on e.g. installing burglar alarms.
- Usually limits set on amount each item can affect premium, and overall limits for all discounts or surcharges.

Retrospective rating

- Premium is retroactively changed by payment of a dividend depending on individuals losses and losses from their risk class.
- Often amount of dividend is only known years afterward, once all claims are settled.

5.2 Increased limits factors

Increased limits factors

- Relative increase in premium caused by increasing policy limit.
- Several difficulties in estimating ILF:
  - Loss development factors increase with policy limit.
  - Trend factors tend to increase with policy limit.
  - Risk to insurer increases faster than expected claim.
  - Some expenses are fixed; some vary with premium; some vary with policy size.
- Historical policy limits affect the data in several ways:
  - Insurance company records generally censor data at policy limits.
  - Limit can impact settlement amounts — e.g. lawyers might aim for policy limit.
  - Adverse selection
- These data issues can be mitigated by only considering policies with limits at least as high as the limit under consideration, provided there is sufficient data.

Question 74

An insurance company has the following data on its policies:

Policy limit	Losses Limited to			
	50,000	100,000	500,000	1,000,000
50,000	10,000			
100,000	34,000	41,000		
500,000	23,000	26,000	31,000	
1,000,000	11,000	12,300	13,400	17,000

Use this data to calculate the ILFs.

5.2 Increased limits factors

Loss Development

- Loss development factors tend to be larger for larger claims.
- They should be estimated from datasets with a single limit.

Trend Factors

- Lower policy limits reduce the effects of inflation.
- Different trend factors should be calculated for each policy limit.
- For higher policy limits, the larger variance and smaller data set can mean estimates are not credible, so data from other policy limits may need to be used.

Risk

- Higher policy limits increase risk more than premium.
- Typically risk load should be increased to compensate for this increased risk.

Question 75

For a certain line of insurance, the loss amount per claim follows an exponential distribution with mean $a\theta$, where a is the exposure. The policy has a limit l, which is currently set at 5θ per unit of exposure. Losses increase by an inflation rate of 10%. Calculate the percentage increase in expected total payments per claim.

Question 76

An insurance company models the number of claims on its policies as following a Poisson distribution with parameter $\lambda = 100$. Losses follow a Pareto distribution with $\alpha = 3$ and $\theta = 10,000$. The policies have a policy limit per claim of \$50,000. The insurer models aggregate losses as following a normal distribution, and sets its total premiums at the 95th percentile of the agregate loss distribution.
(a) Calculate the current risk loading as a percentage of the gross rate.
(b) Calculate risk loading as a percentage of the gross rate if the company increases the policy limit to \$100,000 per claim.

Question 77

An insurance company charges a risk charge equal to the square of the average loss amount, divided by 50,000. It has the following data on a set of claims from policies with limit \$1,000,000.

Interval	No. of claims	Total claimed
$(0, 10,000]$	2,300	6,850,000
$(10,000, 100,000]$	900	13,600,000
$(100,000, 500,000]$	140	19,400,000
$(500,000, 1,000,000]$	25	18,600,000

Calculate the ILF from \$100,000 to \$500,000, and to \$1,000,000.

5.2 Increased limits factors

Expenses

- Expenses tend to be subdivided into fixed costs and costs that vary.
- Some expenses are proportional to premium, other variable expenses will increase non-linearly with premium, e.g. adjustment expenses.

Loss Distributions

- Parametric loss distributions make calculating ILFs easier.
- To fit parametric distributions case reserves should be used for open claims, because time to settlement is not independent of loss size.
- Case reserves from very recent claims can be subjective, so it is often a good idea to ignore data from most recent years.

Why Reinsurance?

- Insurer's exposure is too concentrated (e.g. geographically) leaving insurer vulnerable to catastrophe.
- Insurer has limited ability to absorb large losses.
- Insurer has financial difficulties.
- Insurer wants to increase stability.
- Insurer wants to exit a line of business.

Types of Reinsurance

- Excess of Loss — if a single loss exceeds the attachment point reinsurer will pay the excess (up to a limit).
- Stop-loss — like excess of loss for portfolio. Attachment point and limit often defined in terms of loss ratio.
- Quota share — proportional reinsurance.
- Catastrophe cover — like stop-loss, but relating to losses from a single event (e.g. earthquake, hurricane).

Question 78

An insurance company models its aggregate losses as following an exponential distribution. It can buy stop-loss reinsurance for a loading of 100% of the expected claim. The insurance company sets its premium so that total premiums equal the mean plus one standard deviation of aggregate payments. Find the attachment point for reinsurance that minimises the insurer's premiums.

Notes on Reinsurance

- For proportional reinsurance, reinsurer pays ceding commission to cover the ceding company's expense costs. Otherwise the ceding company's expense loading would be too high.

Ratemaking for Reinsurance

- Treaties on risk-exposed basis (covers losses during time period) or risk-attaching basis (covers policies written during time period).
- Premium based on ceding company's earned premium for risk-exposed treaties, paid premium for risk-attaching treaties.
- Pricing may be by exposure rating (based on industry averages) or experience rating (based on ceding company's experience).
- For excess-of-loss reinsurance, ILFs can be used to calculate expected losses.
- Simulation is important to estimate variability in losses.
- Catastrophe cover priced using long experience period and catastrophe models incorporating meteorological, seismic and engineering data, to assess the likely impact of catastrophes.
- Catastrophe cover is high risk. This is reflected by high loading for profit and contingencies.

Loss Reserving for Reinsurance

- Delays in payment generally longer, because reinsurer deals with larger policies and because of the delay in notifying the reinsurer.
- Common clause for excess-of-loss reinsurance: reinsurer notified when incurred loss estimates exceed 50% of attachment point.
- Ceding company must publish direct (without reinsurance) and net reserves. Ceding company legally responsible for claim payments even if it cannot collect its payments from the reinsurer.
- Usually company calculates, direct, net and ceded reserves to compare the approaches and remove inconsistencies.
- Reinsurer will usually group by treaty type and coverage type. May also group by attachment point.
- Catastrophes are known about quickly after the event, but the total reinsured losses will often take a long time to assess. The sudden demand for repairs, etc. can cause inflation in claim costs.