ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 4

Model Solutions

1. *An insurance company sells tennant's insurance. It estimates that the standard deviation of the aggregate annual claim is \$82 and the mean is \$75.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.1$, $p = 0.95$.)

   The coefficient of variation for aggregate annual claim is $\frac{82}{75}$. For the average of n years of aggregate claims, the coefficient of variation is $\frac{82}{75\sqrt{n}}$. Using $r = 0.1$ and $p = 0.95$, the standard for full credibility is obtained by solving:

$$\begin{aligned}
P\left(\left|\frac{\overline{X}-\mu}{\mu}\right| < 0.1\right) &> 0.95\\
2\Phi\left(\frac{0.1\times 75\sqrt{n}}{82}\right) - 1 &> 0.95\\
\frac{0.1\times 75\sqrt{n}}{82} &> 1.959964\\
n &> \left(\frac{82\times 1.959964}{0.1\times 75}\right)^2\\
&= 459.199458093
\end{aligned}$$

   so 459 years are needed.

   The standard net premium for this policy is \$75. An individual has claimed a total of \$833 in the last 3 years.

   (b) What is the net Credibility premium for this individual, using limited fluctuation credibility?

   The credibility of 3 years of experience is $Z = \sqrt{\frac{3}{459.199458093}} = 0.080827648457$. The premium for this individual is therefore $0.080827648457\times\frac{833}{3} + 0.919172351543\times 75 = \$91.38$.

2. *A liability insurance company classifies companies as high, medium or low risk. Annual claims from high risk companies follow an inverse gamma*

distribution with $\alpha = 2.7$ and $\theta = 8000$. Annual claims from medium risk companies follow an inverse gamma distribution with $\alpha = 2.9$ and $\theta = 2500$. Annual claims from low risk companies follow a Pareto distribution with $\alpha = 3.3$ and $\theta = 1200$. 5% of companies are high risk, 40% are medium risk and 55% are low risk.

(a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen company.

- For a high-risk company, the expected claim is $\frac{8000}{1.7} = 4705.88235294$. The variance is $\frac{8000^2}{1.7^2 \times 0.7} = 31636183.8853$
- For a medium-risk company, the expected claim is $\frac{2500}{1.9} = 1315.78947368$. The variance is $\frac{2500^2}{1.9^2 \times 0.9} = 1923668.82118$
- For a low-risk company, the expected claim is $\frac{1200}{2.3} = 521.739130435$. The variance is $\frac{3.3 \times 1200^2}{2.3^2 \times 1.3} = 690998.982114$

The overall expected claim amount is

$$0.05 \times 4705.88235294 + 0.4 \times 1315.78947368 + 0.55 \times 521.739130435 = 1048.56642886$$

The expected squared claim amount is

$$0.05 \times (4705.88235294^2 + 31636183.8853) + 0.4 \times (1315.78947368^2 + 1923668.82118) + 0.55 \times (521.739130435^2 + 690998.982114) = 4680829.82063$$

The variance of the claim amount is therefore $4680829.82063 - 1048.56642886^2 = 3581338.2649$.

(b) Given that a company's annual claims over the past 3 years are \$1300, \$13,440 and \$3,020, what are the expectation and variance of the company's claims next year?

The likelihood of an inverse gamma distribution is

$$\left(\frac{\theta^\alpha}{\Gamma(\alpha)}\right)^3 x^{-(\alpha+1)} e^{-\frac{\theta}{x}}$$

The likelhood of the above sequence of 3 claims is given by

$$\left(\frac{\theta^\alpha}{\Gamma(\alpha)}\right)^3 (1300 \times 13440 \times 3020)^{-(\alpha+1)} e^{\theta\left(\frac{1}{1300} + \frac{1}{13440} + \frac{1}{3020}\right)}$$
$$= \left(\frac{\theta^\alpha}{\Gamma(\alpha)}\right)^3 52765440000^{-(\alpha+1)} e^{0.001174761358950}$$

The likelihood of a Pareto distribution is

$$\frac{\alpha\theta^\alpha}{(\theta + x)^{\alpha+1}}$$

The likelihood of the above sequence of 3 claims is given by

$$\begin{aligned}
&\frac{\alpha^3\theta^{3\alpha}}{(\theta+100)^{\alpha+1}(\theta+13440)^{\alpha+1}(\theta+3020)^{\alpha+1}}\\
=&\frac{3.3^3 1200^{9.9}}{(2500\times 14640\times 4220)^{4.3}}\\
=&8.46494126734\times 10^{-17}
\end{aligned}$$

- The likelihood of these claims for a high-risk company is

  $$\left(\frac{8000^{2.7}}{\Gamma(2.7)}\right)^3 52765440000^{-3.7}e^{0.00117476135895\times 8000} = 1.969137{\times}10^{-13}$$

- The likelihood of these claims for a medium-risk company is

  $$\left(\frac{2500^{2.9}}{\Gamma(2.9)}\right)^3 52765440000^{-3.9}e^{0.00117476135895\times 2500} = 4.830199{\times}10^{-15}$$

- The likelihood of these claims for a low-risk company is

  $$1.40867099213\times 10^{-15}$$

The posterior probabilites are therefore:

$$\frac{0.05\times 1.969137\times 10^{-13}}{0.05\times 1.969137\times 10^{-13}+0.4\times 4.830199\times 10^{-15}+0.55\times 8.46494126734\times 10^{-17}}$$
$$\frac{0.4\times 4.830199\times 10^{-15}}{0.05\times 1.969137\times 10^{-13}+0.4\times 4.830199\times 10^{-15}+0.55\times 8.46494126734\times 10^{-17}}$$
and
$$\frac{0.55\times 8.46494126734\times 10^{-17}}{0.05\times 1.969137\times 10^{-13}+0.4\times 4.830199\times 10^{-15}+0.55\times 8.46494126734\times 10^{-17}1.40867099213\times 10^{-1}}$$

which gives

0.832663825096, 0.163398767087 and 0.00395297227883.

This means that the expected aggregate claim is

$$0.832663825096\times 4705.88235294+0.163398767087\times 1315.78947368+0.00395297227883\times 521.739130435 = \$4,135.48$$

The expected squared aggreagate claim is

$$\begin{aligned}
0.832663825096\times(4705.88235294^2+31636183.8853)+0.163398767087\times(1315.78947368^2+1923668.82118)\\
+0.00395297227883\times(521.739130435^2+690998.982114) = 45382945.2663
\end{aligned}$$

The variance of aggregate claims is

$$45382945.2663-4135.47879852^2 = 28,280,760.3733$$

Standard Questions

3. *A group health insurance company has the following standards for full credibilty: 480 person-years for frequency and 369 claims for severity.*

   The book estimates are 0.7 claims per person-year for claim frequency, and \$732 per claim for claim severity. Find the single standard in terms of person-years that gives the same premium for

   (a) A company that has 304 claims from 291 person-years, with an average claim amount of \$308.

   The credibility estimate for claim frequency is $\sqrt{\frac{291}{480}}\times\frac{304}{291}+\left(1-\sqrt{\frac{291}{480}}\right)\times 0.7=0.96836990957$. The credibility estimate for claim severity is $\sqrt{\frac{304}{369}}\times 308+\left(1-\sqrt{\frac{304}{369}}\right)\times 732=347.151794962$. Thus the credibility premium is $0.96836990957\times 347.151794962=336.171352294$ for each person-year. Since the book premium is $0.7\times 732=512.4$, and the experience is $\frac{304\times 308}{291}=321.759450172$, a single standard would need credibility given by

$$\begin{aligned}321.759450172Z+512.4(1-Z)&=336.171352294\\ 512.4-190.640549828Z&=336.171352294\\ 190.640549828Z&=176.228647706\\ Z&=0.924402745717\end{aligned}$$

   If n_0 is the number of policy years for full credibility, then this gives

$$\begin{aligned}\sqrt{\frac{291}{n_0}}&=0.924402745717\\ \frac{291}{n_0}&=0.854520436289\\ n_0&=\frac{291}{0.854520436289}\\ &=340.54188483\end{aligned}$$

   (b) A company that has 94 claims from 310 person-years, with an average claim amount of \$958.

   The credibility estimate for claim frequency is $\sqrt{\frac{310}{480}}\times\frac{94}{310}+\left(1-\sqrt{\frac{310}{480}}\right)\times 0.7=0.381137353871$. The credibility estimate for claim severity is $\sqrt{\frac{94}{369}}\times 958+\left(1-\sqrt{\frac{94}{369}}\right)\times 732=846.066780231$. Thus the credibility premium is $0.381137353871\times 846.066780231=322.467653815$ for each person-year. Since the book premium is $0.7 \times 732 = 512.4$, and the experience is $\frac{94\times958}{310} = 290.490322581$, a single standard would need credibility given by

$$\begin{aligned}
290.490322581Z + 512.4(1-Z) &= 322.467653815\\
512.4 - 221.909677419Z &= 322.467653815\\
221.909677419Z &= 189.932346185\\
Z &= 0.855899338839
\end{aligned}$$

If n_0 is the number of policy years for full credibility, then this gives

$$\begin{aligned}
\sqrt{\frac{310}{n_0}} &= 0.855899338839\\
\frac{310}{n_0} &= 0.732563678225\\
n_0 &= \frac{310}{0.732563678225} & &= 423.171403681
\end{aligned}$$

4. *A medical liability insurer classifies doctors as "low-risk" and "high-risk". It estimates that 70% of doctors are low-risk. Annual claims from low-risk doctors are modelled as following a Pareto distribution with $\alpha = 3.8$ and $\theta = 1,340$. Annual claims from high-risk doctors have mean \$935 and variance 1,503,060.*

   It is considering modelling claims for high-risk doctors using either a Pareto distribution or an inverse gamma distribution, with parameters fitted using the method of moments. Which of these distributions would result in a higher Bayes premium for a doctor whose annual claims in the previous two years were \$146 and \$3,632?

   If this doctor is low-risk, then the likelihood of these claims is

$$\frac{3.8^2 \times 1340^{7.6}}{1486^{4.8} \times 4972^{4.8}} = 9.04740433566 \times 10^{-9}$$

For the Pareto distribution, the parameters are given by

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 935\\
\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} &= 1503060\\
\frac{\alpha}{\alpha-2} &= \frac{1503060}{935^2}\\
\frac{2}{\alpha} &= 1-\frac{935^2}{1503060}\\
\alpha &= \frac{3006120}{628835}\\
&= 4.78045910294\\
\theta &= 3534.72926125
\end{aligned}$$

The likelihood of the claims is

$$\frac{4.78045910294^2 \times 3534.72926125^{9.56091820588}}{3680.72926125^{5.78045910294} \times 7166.72926125^{5.78045910294}} = 2.43344703218 \times 10^{-8}$$

For the inverse gamma distribution, the parameters are given by

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 935\\
\frac{\theta}{(\alpha-1)^2(\alpha-2)} &= 1503060\\
\frac{1}{\alpha-2} &= \frac{1503060}{935^2}\\
\alpha &= 2+\frac{935^2}{1503060}\\
&= 2.58163014118\\
\theta &= 1478.824182
\end{aligned}$$

The likelihood of the claims is

$$\frac{1478.824182^{5.16326028236} e^{-1478.824182\left(\frac{1}{146}+\frac{1}{3632}\right)}}{146^{3.58163014118} \times 3632^{3.58163014118} \Gamma(2.58163014118)^2} = 9.76717 \times 10^{-10}$$

Since the Pareto distribution assigns higher likelihood of being high-risk, the posterior probability of being high risk, and therefore the premium, is higher for the Pareto distribution.

[For the Pareto distribution, the posterior probability of being high-risk is

$$\frac{0.3 \times 2.43344703218 \times 10^{-8}}{0.3 \times 2.43344703218 \times 10^{-8} + 0.7 \times 9.04740433566 \times 10^{-9}} = 0.535469848157$$

so the premium is $0.535469848157 \times 935 + 0.464530151843 \times \frac{1340}{2.8} = \$722.98$.

For the inverse gamma distribution, the posterior probability of being high-risk is

$$\frac{0.3 \times 9.76717 \times 10^{-10}}{0.3 \times 9.76717 \times 10^{-10} + 0.7 \times 9.04740433566 \times 10^{-9}} = 0.0442206970018$$

so the premium is $0.0442206970018 \times 935 + 0.955779302998 \times \frac{1340}{2.8} = \$498.76$.

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5. *An insurance company is pricing its policies for fire insurance. It insures 6 buildings, owned by 3 companies. Buildings 1, 2, 3 and 4 are in area A, while buildings 5 and 6 are in area B. Buildings 1, 2, and 5 are owned by company X, buildings 3 and 6 are owned by company Y, and building 4 is owned by company Z. The insurance company is setting a greatest accuracy credibility premium for each building. It will use a formula $\hat{X_{i,2}} = \beta_{i,0} + \sum_{j=1}^{6} \alpha_{ij} X_{j,1}$ where $X_{i,k}$ is the total loss for building i in year j, to calculate the credibility premium. It makes the following modelling assumptions:*

   - *The prior expected aggregate claims and variances of aggregate claims for each building are given in the following table:*

Building	*Expected aggregate claims*	*Variance of aggregate claims*
1	*\$ 4,904*	*483,600*
2	*\$ 2,048*	*301,500*
3	*\$ 3,360*	*289,700*
4	*\$11,421*	*1,004,200*
5	*\$ 9,310*	*1,143,000*
6	*\$ 5,082*	*821,100*

   - *For a given building, the correlation between the losses in different years is 0.45. (Recall $Corr(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$)*
   - *The correlation between aggregate claims for buildings in the same region in different years is 0.32 if owned by the same company, and 0.22 if owned by different companies.*
   - *The correlation between aggregate claims for buidings owned by the same company in different regions is 0.25.*
   - *Aggregate claims for buildings in different regions owned by different companies are independent.*

Find a set of equations which can determine the values of $\beta_{1,0}$, and α_{1j}.
[You do not need to solve these equations.]

We use our standard equations:

$$\mathbb{E}(X_{i,2}) = \beta_{i,0} + \sum_{j=1}^{6} \alpha_i \mathbb{E}(X_{i,1})$$
$$\mathrm{Cov}(X_{i,2}, X_{j,1}) = \sum_{i=1}^{6} \alpha_{ij}\, \mathrm{Cov}(X_{i,2}, X_{j,1})$$

Substituting in the numbers given, these equations become:

$$\begin{aligned}
4904 &= \beta_{1,0} + 4904\alpha_{11} + 2048\alpha_{12} + 3360\alpha_{13} + 11421\alpha_{14} + 9310\alpha_{15} + 5082\alpha_{16}\\
0.45 \times 483600 &= 483600\alpha_{1,1} + 0.32 \times \sqrt{483600 \times 301500}\alpha_{1,2} + 0.22 \times \sqrt{483600 \times 289700}\alpha_{1,3}\\
&\quad + 0.22 \times \sqrt{483600 \times 1004200}\alpha_{1,4} + 0.25 \times \sqrt{483600 \times 1143000}\alpha_{1,5} + 0 \times \sqrt{483600 \times 821100}\alpha_{1,6}\\
0.32 \times \sqrt{483600 \times 301500} &= 0.32 \times \sqrt{301500 \times 483600}\alpha_{1,1} + 301500\alpha_{1,2} + 0.22 \times \sqrt{301500 \times 289700}\alpha_{1,3}\\
&\quad + 0.22 \times \sqrt{301500 \times 1004200}\alpha_{1,4} + 0.25 \times \sqrt{301500 \times 1143000}\alpha_{1,5} + 0 \times \sqrt{301500 \times 821100}\alpha_{1,6}\\
0.22 \times \sqrt{289700 \times 483600} &= 0.22 \times \sqrt{289700 \times 483600}\alpha_{1,1} + 0.22 \times \sqrt{289700 \times 301500}\alpha_{1,2} + 289700\alpha_{1,3} +\\
&\quad 0.22 \times \sqrt{289700 \times 1004200}\alpha_{1,4} + 0 \times \sqrt{289700 \times 1143000}\alpha_{1,5} + 0.25 \times \sqrt{289700 \times 821100}\alpha_{1,6}\\
0.22 \times \sqrt{289700 \times 483600} &= 0.22 \times \sqrt{1004200 \times 483600}\alpha_{1,1} + 0.22 \times \sqrt{1004200 \times 301500}\alpha_{1,2} + 0.22 \times \sqrt{1004200 \times 289700}\alpha_{1,3}\\
&\quad + 1004200\alpha_{1,4} + 0 \times \sqrt{1004200 \times 1143000}\alpha_{1,5} + 0 \times \sqrt{1004200 \times 821100}\alpha_{1,6}\\
0.25 \times \sqrt{1143000 \times 483600} &= 0.25 \times \sqrt{1143000 \times 483600}\alpha_{1,1} + 0.25 \times \sqrt{1143000 \times 301500}\alpha_{1,2} + 0 \times \sqrt{1143000 \times 289700}\alpha_{1,3}\\
&\quad + 0 \times \sqrt{1143000 \times 1004200}\alpha_{1,4} + 1143000\alpha_{1,5} + 0.22 \times \sqrt{1143000 \times 821100}\alpha_{1,6}\\
0 \times \sqrt{821100 \times 483600} &= 0 \times \sqrt{821100 \times 483600}\alpha_{1,1} + 0 \times \sqrt{821100 \times 301500}\alpha_{1,2} + 0.25 \times \sqrt{821100 \times 289700}\alpha_{1,3}\\
&\quad + 0 \times \sqrt{821100 \times 1004200}\alpha_{1,4} + 0.22 \times \sqrt{821100 \times 1143000}\alpha_{1,5} + 821100\alpha_{1,6}
\end{aligned}$$