ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 5

Model Solutions

Basic Questions

1. *An insurance company sets the book pure premium for its liability insurance at $5,410. The expected process variance is 5,859,000 and the variance of hypothetical means is 1,310,600. If a company has aggregate claims of $12,500 over the past 9 years, calculate the credibility premium for this company's next year's insurance using the Bühlmann model.*

   The credibility of 9 years of experience is $Z = \frac{9}{9+\frac{5859000}{1310600}} = 0.668128058728$. The credibility premium for this individual is therefore $0.668128058728 \times \frac{12500}{9} + 0.331871941272 \times 5410 = \$2,723.38$.

2. *An insurance company has the following data on a workers' compensation insurance policy for a company.*

Year	*1*	*2*	*3*	*4*	*5*
Exposure	*4,392*	*5,045*	*4,803*	*5,107*	*5,246*
Aggregate claims	*$423,100*	*$0*	*$746,400*	*$1,062,700*	*$547,300*

   The book premium is $494 per unit of exposure. The variance of hypothetical means per unit of exposure is 351,326. The expected process variance per unit of exposure is 7,926,306,225. Using a Bühlmann-Straub model, calculate the credibility premium for Year 6 if the company has 5,475 units of exposure.

   The company has aggregate claims of

   $2,779,500 from 24,593 units of experience. The credibility of the company's experience is $Z = \frac{24593}{24593+\frac{7926306225}{351326}} = 0.521545152406$. Therefore the company's new premium per unit of exposure is $0.521545152406 \times \frac{2779500}{24593} + 0.478454847594 \times 494 = 295.301709598$. The total premium for this company for 5,475 units of exposure is therefore $5475 \times 295.301709598 = \$1,616,776.86$.

3. *An insurance company has the following previous data on aggregate claims:*

Policyholder	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*	*Mean*	*Variance*
1	*1318.14*	*4535.25*	*1484.57*	*0.00*	*4783.58*	*2424.308*	*4501556.23227*
2	*344.07*	*0.00*	*0.00*	*662.08*	*0.00*	*201.230*	*88566.63170*
3	*0.00*	*0.00*	*0.00*	*21.49*	*0.00*	*4.298*	*92.36402*
4	*1051.40*	*0.00*	*392.33*	*0.00*	*0.00*	*288.746*	*210623.38158*
5	*0.00*	*348.42*	*0.00*	*568.79*	*0.00*	*183.442*	*69165.93092*

Calculate the Bühlmann credibility premium for each policyholder in Year 6.

The book premium is the average of the average claims for each individual, i.e. $\frac{2424.308+201.230+4.298+288.746+183.442}{5} = 620.4048$. The estimated EPV is the average of the variances for the individuals, that is

$$\frac{4501556.23227 + 88566.63170 + 92.36402 + 210623.38158 + 69165.93092}{5} = 974000.908098$$

The variance of the observed means is

$$\frac{(2424.308 - 620.4048)^2 + (201.230 - 620.4048)^2 + (4.298 - 620.4048)^2 + (288.746 - 620.4048)^2 + (183.44}{4}$$

The part of this due to process variance is $\frac{974000.908098}{5} = 194800.18162$. Therefore, the estimated VHM is $1027573.97629 - 194800.18162 = 832773.79467$.

The credibility of 5 years of experience is therefore

$$Z = \frac{5}{5 + \frac{974000.908098}{832773.79467}} = 0.81042709711$$

The premiums are therefore:

Policyholder	Premium
1	$0.81042709711 \times 2424.308 + 0.18957290289 \times 620.4048 = \$2,082.34$
2	$0.81042709711 \times 201.230 + 0.18957290289 \times 620.4048 = \$280.69$
3	$0.81042709711 \times 4.298 + 0.18957290289 \times 620.4048 = \$121.10$
4	$0.81042709711 \times 288.746 + 0.18957290289 \times 620.4048 = \$351.62$
5	$0.81042709711 \times 183.442 + 0.18957290289 \times 620.4048 = \$266.28$

Standard Questions

4. *A workers' compensation insurance company models the number of claims made by a company in a year as a Poisson distribution with mean proportional to their exposure multiplied by a constant that varies between companies. It has the following data from 2020:*

Exposure	*No. of Claims*	*Exposure*	*No. of Claims*	*Exposure*	*No. of Claims*
1424	*102*	*516*	*0*	*344*	*28*
997	*3*	*511*	*1*	*299*	*34*
809	*50*	*461*	*17*	*298*	*0*
593	*0*	*425*	*0*	*292*	*22*
589	*0*	*403*	*41*	*246*	*2*
573	*0*	*393*	*0*	*233*	*0*
525	*20*	*381*	*18*	*178*	*0*

Using this data, calculate the credibility estimate for the expected claim frequency in the following year, for the first company, which made 102 claims from 1424 units of exposure, if that company has 1,330 units of exposure the following year.

There were a total of 338 claims from 10490 units of exposure, so the mean claim frequency is $\frac{338}{10490} = 0.0322211630124$ per unit of exposure. For the Poisson distribution, this is also the average EPV per unit of exposure.

The estimator for variance of the hypothetical means is

$$\frac{\sum_{i=1}^{21} m_i \left(\frac{X_i}{m_i} - 0.0322211630124\right)^2 - 20\hat{\sigma}}{\sum_{i=1}^{21} m_i - \frac{\sum_{i=1}^{21} {m_i}^2}{\sum_{i=1}^{21} m_i}}$$

This gives 0.001332394 as the estimated VHM. The credibility of 1424 units of exposure is therefore

$$Z = \frac{1424}{1424 + \frac{0.0322211630124}{0.001332394}} = 0.983301205278$$

We calculate the book premium as the credibility-weighted average of the experience of the companies.

We calculate the credibility-weighted average frequency as 0.03056618141. Thus the expected claim frequency for the first company is

$$1330\left(0.983301205278 \times \frac{102}{1424} + 0.016698794722 \times 0.03056618141\right) = \$94.3548687519$$

Using equal weights for all policyholders:

If we weight all policyholders equally, the average frequency per unit of exposure is

$$\frac{1}{21}\left(\frac{102}{1424} + \frac{3}{997} + \frac{50}{809} + \frac{0}{593} + \frac{0}{589} + \frac{0}{573} + \frac{20}{525} + \frac{0}{516} + \frac{1}{511} + \frac{17}{461} + \frac{0}{425} + \frac{41}{403} + \frac{0}{393} + \frac{18}{381} + \frac{28}{344} + \frac{34}{299} + \frac{0}{298} + \frac{22}{292} + \frac{2}{246} + \frac{0}{233} + \frac{0}{178}\right) = 0.0305206103044$$

For the Poisson distribution, this is also the average EPV per unit of exposure.

The variance of the observed means is

$$\frac{1}{20}\left(\left(\frac{102}{1424}-0.0305206\right)^2+\left(\frac{3}{997}-0.0305206\right)^2+\left(\frac{50}{809}-0.0305206\right)^2+\left(\frac{0}{593}-0.0305206\right)^2\right.$$
$$+\left(\frac{0}{589}-0.0305206\right)^2+\left(\frac{0}{573}-0.0305206\right)^2+\left(\frac{20}{525}-0.0305206\right)^2+\left(\frac{0}{516}-0.0305206\right)^2$$
$$+\left(\frac{1}{511}-0.0305206\right)^2+\left(\frac{17}{461}-0.0305206\right)^2+\left(\frac{0}{425}-0.0305206\right)^2+\left(\frac{41}{403}-0.0305206\right)^2$$
$$+\left(\frac{0}{393}-0.0305206\right)^2+\left(\frac{18}{381}-0.0305206\right)^2+\left(\frac{28}{344}-0.0305206\right)^2+\left(\frac{34}{299}-0.0305206\right)^2$$
$$+\left(\frac{0}{298}-0.0305206\right)^2+\left(\frac{22}{292}-0.0305206\right)^2+\left(\frac{2}{246}-0.0305206\right)^2+\left(\frac{0}{233}-0.0305206\right)^2$$
$$\left.+\left(\frac{0}{178}-0.0305206\right)^2\right)=0.0015046804576$$

Of this,

$$\frac{0.0305206}{21}\left(\frac{1}{1424}+\frac{1}{997}+\frac{1}{809}+\frac{1}{593}+\frac{1}{589}+\frac{1}{573}+\frac{1}{525}+\frac{1}{516}+\frac{1}{511}+\frac{1}{461}+\frac{1}{425}+\frac{1}{403}+\frac{1}{393}\right.$$
$$\left.+\frac{1}{381}+\frac{1}{344}+\frac{1}{299}+\frac{1}{298}+\frac{1}{292}+\frac{1}{246}+\frac{1}{233}+\frac{1}{178}\right)=0.0000771005720761$$

is due to process variance. Thus, the estimated VHM is $0.0015046804576-0.0000771005720761=0.00142757988552$. The credibility of 1424 units of exposure is therefore

$$Z=\frac{1424}{1424+\frac{0.0305206103044}{0.00142757988552}}=0.985208542394$$

We calculate the book premium as the credibility-weighted average of the experience of the companies which gives 0.0290113803253.

Thus the expected claim frequency for the first company is

$$1330\left(0.985208542394\times\frac{102}{1424}+0.014791457606\times 0.0290113803253\right)=94.4284487022$$

5. *Aggregate claims for a given individual policy are modelled as following an inverse gamma distribution with $\alpha=5$ and θ varying between individuals. The first 5 years of experience on this policy are:*

Policyholder	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*	*Mean*	*Variance*
1	*87.11*	*80.68*	*0.00*	*43.41*	*0.00*	*42.240*	*1765.18865*
2	*15.06*	*0.00*	*0.00*	*0.00*	*0.00*	*3.012*	*45.36072*
3	*79.61*	*231.22*	*140.19*	*0.00*	*0.00*	*90.204*	*9692.46713*
4	*0.00*	*23.44*	*22.07*	*51.07*	*165.67*	*52.450*	*4334.04995*

(a) Estimate the EPV and VHM.

The mean claim amount is $\frac{42.240+3.012+90.204+52.450}{4} = 46.9765$. The mean of the inverse gamma distribution is $\frac{\Theta}{4}$, while the variance is $\frac{\Theta^2}{48}$. Thus, the EPV is $\frac{1}{48}\mathbb{E}(\Theta^2) = \frac{1}{48}\left(\mathbb{E}(\Theta)^2 + \operatorname{Var}(\Theta)\right)$. Meanwhile, the hypothetical means are $\frac{\Theta}{4}$, so the VHM is $\operatorname{Var}\left(\frac{\Theta}{4}\right) = \frac{1}{16}\operatorname{Var}(\Theta)$. We have that $\frac{1}{4}\mathbb{E}(\Theta) = 46.9765$, so $\mathbb{E}(\Theta) = 187.906$. The variance of observed means is

$$\text{VHM} + \frac{\text{EPV}}{5} = \frac{1}{16}\operatorname{Var}(\Theta) + \frac{1}{5\times 48}\left(187.906^2 + \operatorname{Var}(\Theta)\right)$$

From the data, the estimated variance of observed means is

$$\frac{1}{3}\left((42.240-46.9765)^2 + (3.012-46.9765)^2 + (90.204-46.9765)^2 + (52.450-46.9765)^2\right) = 1284.629217$$

Thus we have

$$\begin{aligned}\frac{1}{16}\operatorname{Var}(\Theta) + \frac{1}{5\times 48}\left(187.906^2 + \operatorname{Var}(\Theta)\right) &= 1284.629217\\ 16\operatorname{Var}(\Theta) + 187.906^2 &= 240 \times 1284.629217\\ \operatorname{Var}(\Theta) &= 17062.6467028\end{aligned}$$

Thus, the EPV is

$$\frac{1}{48}\left(187.906^2 + 17062.6467028\right) = 1091.06899039$$

and the VHM is

$$\frac{17062.6467028}{16} = 1066.41541893$$

(b) Calculate the credibility premium for Policyholder 3 in the next year.

The credibility of 5 years' of experience is

$$Z = \frac{5}{5 + \frac{1091.06899039}{1066.41541893}} = 0.830134800618$$

Thus the premium for Policyholder 3 is

$$0.830134800618 \times 90.204 + 0.169865199382 \times 46.9765 = \$82.86$$