ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 1

Model Solutions

Basic Questions

1. *Aggregate payments have a compound distribution. The frequency distribution is negative binomial with $r = 3$ and $\beta = 0.5$. The severity distribution is gamma with shape $\alpha = 2.3$ and scale $\theta = 400$. Use a gamma approximation to aggregate payments to estimate the probability that aggregate payments are more than 4,000.*

   The frequency distribution has mean $3 \times 0.5 = 1.5$ and variance $3 \times 0.5 \times 1.5 = 2.25$. The severity distribution has mean $400 \times 2.3 = 920$ and variance $400^2 \times 2.3 = 368000$.

   The mean of aggregate losses is given by $1.5 \times 920 = 1380$ and variance $1.5 \times 368000 + 2.25 \times 920^2 = 2456400$. Setting these equal to the mean and variance of a gamma distribution with parameters α and θ gives

$$\begin{aligned}\alpha\theta &= 1380\\ \alpha\theta^2 &= 2456400\\ \alpha &= \frac{1380^2}{2456400} = 0.775280898876\\ \theta &= \frac{1380}{0.775280898876} = 1780\end{aligned}$$

   For these parameters, the probability that payments exceed \$4,000 is

```
pgamma(4000/1780,shape=0.775280898876,lower.tail=FALSE)
```

   This probability is 0.06868348.

2. *Loss amounts follow a gamma distribution with shape $\alpha = 1.3$ and scale $\theta = 1500$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.930*
1	*0.024*
2	*0.015*
3	*0.031*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above \$5,000. Calculate the expected payment for this excess-of-loss reinsurance.

If there are n claims, then the total losses follow a gamma distribution with shape $\alpha = 1.3n$ and $\theta = 1500$. The expected payment on the excess of loss distribution in this case is therefore given by:

$$\begin{aligned}\mathbb{E}((X-5000)_+) &= \frac{1500}{\Gamma(1.3n)}\int_{\frac{5000}{1500}}^{\infty}\left(x-\frac{5000}{1500}\right)x^{1.3n-1}e^{-x}\,dx\\ &= \frac{1500}{\Gamma(1.3n)}\left(\int_{\frac{5000}{1500}}^{\infty}x^{1.3n-1}e^{-x}\,dx - \frac{5000}{1500}\int_{\frac{5000}{1500}}^{\infty}x^{1.3n-1}e^{-x}\,dx\right)\\ &= 1500\,(1.3n\texttt{pgamma(5000/1500,shape=1.3n+1,lower.tail=FALSE)}\\ &\qquad -5000/1500\texttt{pgamma(2,shape=1.3n,lower.tail=FALSE)})\end{aligned}$$

This gives the following table

n	$P(N=n)$	$\mathbb{E}((S-5000)_+\mid N=n)$	$\mathbb{E}((S-5000)_+ I_{N=n})$
0	0.930	0.000	0.000
1	0.024	97.8740	2.348976
2	0.015	562.7611	8.4414165
3	0.031	1574.3086	48.8035666

So the total expected payment is $2.348976 + 8.4414165 + 48.8035666 =$ \$59.59.

3. *Claim frequency follows a negative binomial distribution with* $r = 4.8$ *and* $\beta = 1.2$. *Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
1	*0.24*
2	*0.30*
3	*0.26*
4 or more	*0.20*

Use the recursive method to calculate the exact probability that aggregate claims are at least \$4,000.

Since the severity distribution is zero-truncated, the aggregate loss distribution is zero only if claim frequency is zero, which has probability

$\frac{1}{(1+1.2)^{4.8}} = 0.0227180542376$. Recall that for the negative binomial distribution, $a = \frac{\beta}{1+\beta} = \frac{1.2}{2.2}$ and $b = \frac{(r-1)\beta}{1+\beta} = \frac{4.56}{2.2}$.

The recurrence formula is

$$f(x) = \sum_{k=1}^{x} \left(\frac{1.2}{2.2} + \frac{4.56k}{2.2x}\right) f_X(k) f(x-k)$$

Applying this gives:

$$f(1) = \left(\frac{1.2}{2.2} + \frac{4.56}{2.2}\right) \times 0.24 \times 0.0227180542376 = 0.0142751991719$$
$$f(2) = \left(\frac{1.2}{2.2} + \frac{2.28}{2.2}\right) \times 0.24 \times 0.0142751991719 + \left(\frac{1.2}{2.2} + \frac{4.56}{2.2}\right) \times 0.30 \times 0.0227180542376 = 0.0232633836687$$
$$f(3) = \left(\frac{1.2}{2.2} + \frac{1.52}{2.2}\right) \times 0.24 \times 0.0232633836687 + \left(\frac{1.2}{2.2} + \frac{3.04}{2.2}\right) \times 0.30 \times 0.0142751991719 + \left(\frac{1.2}{2.2} + \frac{4.56}{2.2}\right) \times 0.26 \times 0.022718054$$

The probability that aggregate claims are at least \$4,000 is therefore

$$\begin{aligned}&1 - f(0) - f(1) - f(2) - f(3)\\ =&1 - 0.0227180542376 - 0.0142751991719 - 0.0232633836687 - 0.0306213401055\\ =&0.909122022816\end{aligned}$$

4. *Use an arithmetic distribution ($h = 1$) to approximate a Pareto distribution distribution with shape $\alpha = 3.5$ and scale $\theta = 6.6$.*

(a) Using the method of rounding, calculate the mean of the arithmetic approximation. [You can evaluate this numerically: use 5,000 terms in the sum.]

Using the method of rounding, we set $p_0 = P\left(X < \frac{1}{2}\right) = 1 - \left(\frac{6.6}{7.1}\right)^{3.5} = 0.225538997744$ and $p_n = P\left(n - \frac{1}{2} < X < n + \frac{1}{2}\right) = \left(\frac{6.6}{6.1+n}\right)^{3.5} - \left(\frac{6.6}{7.1+n}\right)^{3.5}$ for $n > 0$. This gives

$$\begin{aligned}\mathbb{E}(X_a) &= \sum_{n=1}^{\infty} P(X_a \geqslant n)\\ &= \sum_{n=1}^{\infty} P\left(X > n - \frac{1}{2}\right)\\ &= \sum_{n=1}^{\infty} \left(\frac{6.6}{6.1+n}\right)^{3.5}\end{aligned}$$

We compute this in `R`.

This gives $\mathbb{E}(X_a) = 2.61826$.

[For comparison, $\mathbb{E}(X) = \frac{6.6}{2.5} = 2.64$.]

(b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value is larger than 3.5.

We have

$$p_0 + p_1 + p_2 + p_{3,l} = 1 - \left(\frac{6.6}{9.6}\right)^{3.5} = 0.730564721926$$

and

$$\begin{aligned}
p_{3,u} + p_{4,l} &= \left(\frac{6.6}{9.6}\right)^{3.5} - \left(\frac{6.6}{10.6}\right)^{3.5} = 0.078962355322 \\
3p_{3,u} + 4p_{4,l} &= \int_3^4 -x\frac{d}{dx}\left(\frac{6.6}{6.6+x}\right)^{3.5} dx \\
&= \left[-x\left(\frac{6.6}{6.6+x}\right)^{3.5}\right]_3^4 + \int_3^4 \left(\frac{6.6}{6.6+x}\right)^{3.5} dx \\
&= 3\left(\frac{6.6}{9.6}\right)^{3.5} - 4\left(\frac{6.6}{10.6}\right)^{3.5} + \int_{9.6}^{10.6} 6.6^{3.5}t^{-3.5}\,dt \\
&= 3\left(\frac{6.6}{9.6}\right)^{3.5} - 4\left(\frac{6.6}{10.6}\right)^{3.5} + \left[-\frac{6.6^{3.5}t^{-2.5}}{2.5}\right]_{9.6}^{10.6} dt \\
&= 3\left(\frac{6.6}{9.6}\right)^{3.5} - 4\left(\frac{6.6}{10.6}\right)^{3.5} + 0.4\left(\frac{6.6^{3.5}}{9.6^{2.5}} - \frac{6.6^{3.5}}{10.6^{2.5}}\right) = 0.273440418538
\end{aligned}$$

So

$$p_{3,u} = 4 \times 0.078962355322 - 0.273440418538 = 0.04240900275$$

Thus, $P(X_a > 3.5) = 1 - 0.730564721926 - 0.04240900275 = 0.227026275324$

Standard Questions

5. *An insurance company models loss frequency as negative binomial with* $r = 3$ *and unknown* β*, and loss severity as gamma with shape* $\alpha = 0.6$ *and scale* $\theta = 2400$*. There is a per-loss deductable of \$500 for the policy.*

   A reinsurance company models aggregate losses using a Pareto distribution with parameters fitted using the method of moments. Using this model, they calculate the cost of stop-loss reinsurance with attachment point \$10,000 and loading of 20% as \$4,000. What is the value of β*?*

[You should get an equation for β, which can easily be solved by a grid-search (calculating a large number of values to find the correct one).]

With a deductible of \$500, the expected payment for each loss is

$$\begin{aligned}\frac{2400}{\Gamma(0.6)}\int_{\frac{500}{2400}}^{\infty}\left(x-\frac{500}{2400}\right)x^{-0.4}e^{-x}\,dx &= \frac{2400}{\Gamma(0.6)}\left(\int_{\frac{500}{2400}}^{\infty}x^{0.6}e^{-x}\,dx-\frac{500}{2400}\int_{\frac{500}{2400}}^{\infty}x^{-0.4}e^{-x}\,dx\right)\\ &= 2400\times 0.6\texttt{pgamma(500/2400,shape=1.6,lower.tail=FALSE)}\\ &\qquad -500\texttt{pgamma(500/2400,shape=0.6,lower.tail=FALSE)}\\ &= 1070.191\end{aligned}$$

The expected squared payment for each loss is

$$\begin{aligned}\frac{2400^2}{\Gamma(0.6)}\int_{\frac{500}{2400}}^{\infty}\left(x-\frac{500}{2400}\right)^2x^{-0.4}e^{-x}\,dx &= \frac{2400^2}{\Gamma(0.6)}\left(\int_{\frac{500}{2400}}^{\infty}x^{1.6}e^{-x}\,dx-2\times\frac{500}{2400}\int_{\frac{500}{2400}}^{\infty}x^{0.6}e^{-x}\,dx+\left(\frac{500}{2400}\right)^2\int_{\frac{500}{2400}}^{\infty}x^{-0.4}e^{-x}\,dx\right)\\ &= 2400^2\times 0.6\times 1.6\texttt{pgamma(500/2400,shape=2.6,lower.tail=FALSE)}\\ &\qquad -2\times 500\times 2400\times 0.6\texttt{pgamma(500/2400,shape=2.6,lower.tail=FALSE)}\\ &= 4288874\end{aligned}$$

Thus the expected aggregate loss is $3\beta\times 1070.191 = 3210.573\beta$ and the variance of the aggregate loss is $3\beta\times 4288874+3\beta(\beta+1)\times 1070.191^2 = 3435926.32944\beta^2+16302548.3294\beta$.

The parameters for the Pareto distribution are given by solving

$$
\begin{aligned}
\frac{\theta}{\alpha-1} &= 3210.573\beta \\
\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)} &= 3435926.32944\beta^2 + 16302548.3294\beta \\
\frac{\alpha}{\alpha-2} &= \frac{3435926.32944\beta^2 + 16302548.3294\beta}{3210.573^2\beta^2} \\
&= 0.33333333333 + \frac{1.58157720959}{\beta} \\
\frac{2}{\alpha} &= 1 - \frac{\beta}{0.33333333333\beta + 1.58157720959} \\
&= \frac{-0.666666666666667\beta + 1.58157720959}{0.33333333333\beta + 1.58157720959} \\
\alpha &= \frac{0.33333333333\beta + 1.58157720959}{0.790788604795 - 0.333333333333\beta} \\
\alpha - 1 &= \frac{0.6666666667\beta + 0.790788604795}{0.790788604795 - 0.333333333333\beta} \\
\theta &= 3210.573\beta\frac{0.6666666667\beta + 0.790788604795}{0.790788604795 - 0.333333333333\beta}
\end{aligned}
$$

The expected payment on the reinsurance with attachment point $a = 10000$ is

$$
\begin{aligned}
&\int_a^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx \\
&= \int_{a+\theta}^\infty \theta^\alpha u^{-\alpha}\,dx \\
&= \frac{\theta^\alpha}{(\alpha-1)(a+\theta)^{\alpha-1}}
\end{aligned}
$$

Thus we have

$$\frac{\theta^\alpha}{(\alpha-1)(10000+\theta)^{\alpha-1}} = 4000$$

$$\frac{\theta}{\alpha-1}\left(\frac{\theta}{10000+\theta}\right)^{\alpha-1} = 4000$$

$$3210.573\beta\left(\frac{3210.573\beta\frac{0.6666666667\beta+0.790788604795}{0.790788604795-0.333333333333\beta}}{10000+3210.573\beta\frac{0.6666666667\beta+0.790788604795}{0.790788604795-0.333333333333\beta}}\right)^{\frac{0.6666666667\beta+0.790788604795}{0.790788604795-0.333333333333\beta}} = 4000$$

$$3210.573\beta\left(\frac{2140.38200011\beta^2+2538.88454326\beta}{(7907.88604795-3333.33333333\beta)+(2140.38200011\beta^2+2538.88454326\beta)}\right)^{\frac{0.6666666667\beta+0.790788604795}{0.790788604795-0.333333333333\beta}} = 4000$$

$$3210.573\beta\left(\frac{2140.38200011\beta^2+2538.88454326\beta}{7907.88604795-794.44879007\beta+2140.38200011\beta^2}\right)^{\frac{0.6666666667\beta+0.790788604795}{0.790788604795-0.333333333333\beta}} = 4000$$

Numerically solving this by a grid search gives $\beta = 3.33337$.

```
beta<-seq_len(10000)/1000
HW1Q5a<-3210.573 *beta*(( 2140.38200011*beta^2 + 2538.88454326*beta)/(7907.88604795 - 79
0.790788604795)/(0.790788604795 - 0.333333333333*beta))-4000
sum(HW1Q5a<0)
beta[3333]
beta<-3.33+seq_len(10000)/1000000
HW1Q5a<-3210.573 *beta*(( 2140.38200011*beta^2 + 2538.88454326*beta)/(7907.88604795 - 79
0.790788604795)/(0.790788604795 - 0.333333333333*beta))-4000
sum(HW1Q5a<0)
beta[3368]
```

6. *The number of claims an insurance company receives follows a negative binomial distribution with $r = 68$ and $\beta = 1.6$. Claim severity follows a negative binomial distribution with $r = 7.2$ and $\beta = 12$. Calculate the probability that aggregate losses exceed \$12,000.*

   (a) Starting the recurrence 6 standard deviations below the mean [You need to calculate 15,000 terms of the recurrence for f_s.]

   Claim frequency has mean $68 \times 1.6 = 108.8$ and variance $68 \times 1.6 \times 2.6 = 282.88$. Claim severity has mean $7.2 \times 12 = 86.4$ and variance $7.2 \times 12 \times 13 = 1123.2$. Aggregate losses therefore have mean $108.8 \times 86.4 = 9400.32$ and variance $108.8 \times 1123.2 + 282.88 \times 86.4^2 = 2233892.0448$. This means that 6 standard deviations below the mean is $9400.32 - 6\sqrt{2233892.0448} = 432.59377686$ We therefore start the recurrence at $x = 432$.

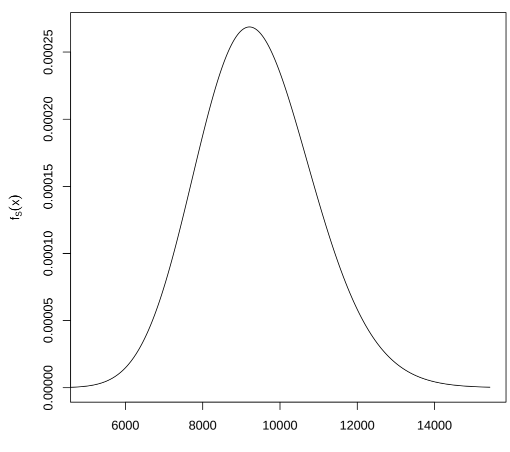
   For the negative binomial distribution with $r = 68$ and $\beta = 1.6$, we have $a = \frac{\beta}{1+\beta} = \frac{1.6}{2.6} = \frac{8}{13}$ and $b = (r-1)\frac{\beta}{1+\beta} = 67 \times \frac{8}{13} = \frac{536}{13}$. The recurrence is therefore

$$f_S(x) = \frac{1}{1-\frac{8}{13}f_X(0)}\sum_{y=1}^{x}\frac{8}{13}\left(1+\frac{67y}{x}\right)f_X(y)f_S(x-y)$$
$$= \frac{1}{1-\frac{8}{13}\times 13^{-7.2}}\sum_{y=1}^{x}\frac{13}{18}\left(1+\frac{67y}{x}\right)13^{-7.2}\binom{y+6.2}{y}\left(\frac{12}{13}\right)^{y}f_S(x-y)$$

```
n<-seq_len(20000)
fx<-choose(n+6.2,n)*(12/13)^n/13^(7.2)
#define a vector of the secondary distribution.

fs<-n #prepare a vector to store results
      # fs[1]=1 is the correct starting value.
for(i in n[-1]-1){#for i from 1 to 19999
     y<-seq_len(i)
     x<-i+432
     fs[i+1]<-sum((1+67*y/x)*fx[y]*fs[i+1-y])*8/(13-8*13^(-7.2))
}
fs<-fs/sum(fs)

# Now fs[i]=fs(1927+i)
Q6a_ans<-sum(fs[(12001-432):20000])
#question asks for strict inequality.
```



This gives the probability that $S > 12000$ as 0.04779442

(b) Using a suitable convolution.

If $N \sim NB(68, 2.6)$, we can say $N = N_1 + N_2 + N_3 + N_4$ with $N_i \sim B(17, 2.6)$. This gives $S = S_1 + \cdots + S_4$, where $S_1 = X_1 + \cdots + X_{N_1}$ and so on. We therefore compute the distribution of each S_i using the recurrence:

$$f_{S_i}(x) = \frac{1}{1 - \frac{8}{13} \times 13^{-7.2}} \sum_{y=1}^{x} \frac{8}{13}\left(1 + \frac{16y}{x}\right) 13^{-7.2} \binom{y+6.2}{y} \left(\frac{12}{13}\right)^y f_S(x-y)$$

We calculate $f_{S_i}(0) = P_{N_i}(f_X(0)) = \left(2.6 - 1.6 \times 13^{-7.2}\right)^{-17} = 8.81968608688 \times 10^{-8}$.

```
g<-rep(0,12001)
g[1]=(2.6-1.6/13^(7.2))^(-17) #f_{S_i}(0)
n<-seq_len(12001)
fx<-choose(n+6.2,n)*(12/13)^n/13^(7.2)


for(x in seq_len(12000)){
    y<-seq_len(x)
    temp<-sum((1+16*y/x)*fx[y]*g[x+1-y])
    g[x+1]<-temp*8/(13-8/13^7.2)
}

ConvolveSelf<-function(n){
  convolution<-vector("numeric",2*length(n))
  for(i in 1:(length(n))){
    convolution[i]<-sum(n[1:i]*n[i:1])
  }
  for(i in 1:(length(n))){
    convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)]*n[length(n)+1-(i:1)])
  }
  return(convolution)
}

g2<-ConvolveSelf(g)
g4<-ConvolveSelf(g2)

Q6b_ans<-1-sum(g4[seq_len(12001)])
# remember the indices of g4 are offset by 1 so that the first index is f_S(0).
```

This gives the probability that $S > 12000$ as 0.04773614.

[The maximum difference in estimated probabilies between these two methods is 4.973579×10^{-13} for $x = 9210$. The first method is faster, taking

19.076 seconds on my computer, while the second method takes 60.582 seconds.]