MATH/STAT 4703, Actuarial Models II
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Toby Kenney
Formula Sheet

General Mathematics

- Quadratic Formula: Solution to $ax^2+bx+c=0$ is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- Gamma function: $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$ satisfies $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

Moments

Centralised moments in terms of uncentralised moments:

$$\mu_2=\mu_2'-\mu^2$$
$$\mu_3=\mu_3'-3\mu\mu_2'+2\mu^3$$
$$\mu_4=\mu_4'-4\mu\mu_3'+6\mu^2\mu_2'-3\mu^4$$

Risk Measures

- Standard deviation principle $r=\mu+a\sigma$.
- Value at Risk $r=\pi_p$.
- Tail Value at Risk $r=\frac{\int_{\pi_p}^\infty xf(x)dx}{1-p}$
$$=\pi_p+\frac{\int_{\pi_p}^\infty S(x)dx}{1-p}$$

Continuous Distributions: Transformed Beta family

Transformed Beta

Inverse of Transformed Beta with $\alpha=\tau$, $\tau=\alpha$, $\theta=\frac{1}{\theta}$.

Support	$[0,\infty)$
Density function	$f(x)=\left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right)\frac{\gamma\left(\frac{x}{\theta}\right)^{\tau\gamma}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+\tau}}$
Mean	$\theta\frac{\Gamma\left(\tau+\frac{1}{\gamma}\right)\Gamma\left(\alpha-\frac{1}{\gamma}\right)}{\Gamma(\tau)\Gamma(\alpha)}$
Raw Moments	$\mu_k'=\theta^k\frac{\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\tau)\Gamma(\alpha)}$
Moment Generating Function	Undefined

Burr

Transformed Beta with $\tau=1$.

Support	$[0,\infty)$
Density function	$f(x)=\frac{\alpha\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$
Mean	$\theta\frac{\Gamma\left(\alpha-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma(\alpha)}$
Raw Moments	$\mu_n'=\theta^n\frac{n\Gamma\left(\alpha-\frac{n}{\gamma}\right)\Gamma\left(\frac{n}{\gamma}\right)}{\Gamma(\alpha)}$
Moment Generating Function	Undefined

Inverse Burr

Transformed Beta with $\alpha=1$.

Density function	$f(x)=\frac{\tau\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\tau+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$
Mean	$\theta\frac{\Gamma\left(\tau+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)}{\Gamma(\tau)}$
Raw Moments	$\mu_k'=\theta^k\frac{\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)}{\Gamma(\tau)}$
Moment Generating Function	Undefined

Generalised Pareto

Transformed Beta with $\gamma=1$.

Support	$[0,\infty)$
Density function	$f(x)=\left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right)\frac{\left(\frac{x}{\theta}\right)^\tau}{x\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha+\tau}}$
Mean	$\theta\frac{\tau}{\alpha-1}$
Raw Moments	$\mu_k'=\theta^k\frac{\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\tau)\Gamma(\alpha)}$
Moment Generating Function	Undefined

Pareto

Transformed Beta with $\tau = \gamma = 1$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\alpha}{\theta\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha+1}} = \frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)\right)^\alpha} = \left(\frac{\theta}{\theta+x}\right)^\alpha$
Mean	$\frac{\theta}{\alpha-1}$ (if $\alpha > 1$)
Variance	$\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha > 2$)
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma(1+k)\Gamma(\alpha-k)}{\Gamma(\alpha)}$
Moment Generating Function	Undefined

Inverse Pareto

Transformed Beta with $\alpha = \gamma = 1$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)}{x\left(1+\left(\frac{\theta}{x}\right)\right)^{\tau+1}}$
Survival Function	$1 - \frac{1}{\left(1+\left(\frac{\theta}{x}\right)\right)^\tau}$
Mean	undefined
Moment Generating Function	Undefined

log-logistic

Transformed Beta with $\alpha = \tau = 1$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^2}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)}$
Mean	$\theta\Gamma\left(1+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)$
Raw Moments	$\mu'_k = \theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)$
Moment Generating Function	Undefined

Paralogistic

Transformed Beta with $\tau = 1$, $\alpha = \gamma$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\gamma+1}}$
Survival function	$S(x) = \frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\gamma}$
Mean	$\theta\frac{\Gamma\left(\gamma-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma(\gamma)}$
Raw Moments	$\mu'_k = \theta^k\frac{\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(\gamma-\frac{k}{\gamma}\right)}{\Gamma(\gamma)}$
Variance	
Moment Generating Function	Undefined

Inverse Paralogistic

Transformed Beta with $\alpha = 1$, $\tau = \gamma$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\gamma\left(\frac{\theta}{x}\right)^\gamma}{x\left(1+\left(\frac{\theta}{x}\right)^\gamma\right)^{\gamma+1}}$
Survival function	$S(x) = 1 - \frac{1}{\left(1+\left(\frac{\theta}{x}\right)^\gamma\right)^\gamma}$
Mean	
Raw Moments	$\mu'_k = \theta^k\frac{\Gamma\left(\gamma+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)}{\Gamma(\gamma)}$
Variance	
Excess loss	
Moment Generating Function	Undefined

Continuous Distributions: Transformed Gamma family

Transformed Gamma

Limit of Transformed Beta as $\alpha \to \infty$ and $\theta \to \infty$ with $\alpha\theta^\alpha = \xi$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\tau\alpha}e^{-\left(\frac{x}{\theta}\right)^\tau}}{x\Gamma(\alpha)}$
Mean	$\mu = \theta\frac{\tau\left(\alpha+\frac{1}{\tau}\right)}{\tau(\alpha)}$
Raw moments	$\mu'_n = \theta^n\frac{\Gamma\left(\alpha+\frac{n}{\tau}\right)}{\Gamma(\alpha)}$

Gamma

Transformed Gamma with $\tau = 1$

Support	$[0, \infty)$
Density function	$f(x) = \frac{\left(\frac{x}{\theta}\right)^\alpha e^{-\left(\frac{x}{\theta}\right)}}{x\Gamma(\alpha)}$
Survival function (for $\alpha \in \mathbb{Z}^+$)	$S(x) = e^{-\frac{x}{\theta}}(1 + \cdots + \frac{\left(\frac{x}{\theta}\right)^{\alpha-1}}{(\alpha-1)!})$
Mean	$\mu = \theta\alpha$
Raw moments	$\mu_n' = \theta^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$
Variance	$\mu_n = \theta^n \alpha$
Moment Generating Function	$M(t) = \frac{1}{(1-\theta t)^\alpha}$

Weibull

Transformed Gamma with $\alpha = 1$

Support	$[0, \infty)$
Density function	$f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^\tau e^{-\left(\frac{x}{\theta}\right)^\tau}}{x}$
Survival function	$e^{-\left(\frac{x}{\theta}\right)^\tau}$
Mean	$\mu = \theta\Gamma\left(1 + \frac{1}{\tau}\right)$
Raw moments	$\mu_n' = \theta^n\Gamma\left(1 + \frac{n}{\tau}\right)$

Exponential

Transformed Gamma with $\alpha = \tau = 1$

Support	$[0, \infty)$
Density function	$f(x) = \frac{e^{-\left(\frac{x}{\theta}\right)}}{\theta}$
Survival function	$e^{-\frac{x}{\theta}}$
Mean	$\mu = \theta$
Raw moments	$\mu_n' = n!\theta^n$
Variance	$\mu_n = \theta^n$
Excess loss	$\theta e^{-\frac{x}{\theta}}$
Moment Generating Function	$M(t) = \frac{1}{1-\theta t}$

Inverse Transformed Gamma

Inverse of transformed gamma with $\theta = \frac{1}{\theta}$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\tau\alpha} e^{-\left(\frac{\theta}{x}\right)^\tau}}{x\Gamma(\alpha)}$
Mean	$\mu = \theta\frac{\Gamma\left(\alpha-\frac{1}{\tau}\right)}{\Gamma(\alpha)}$ (if $\tau\alpha > 1$)
Raw moments	$\mu_n' = \theta^n\frac{\Gamma\left(\alpha-\frac{n}{\tau}\right)}{\Gamma(\alpha)}$ (if $\tau\alpha > n$)

Inverse Gamma

Inverse Transformed Gamma with $\tau = 1$.
Inverse of gamma distribution with $\theta = \frac{1}{\theta}$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\left(\frac{\theta}{x}\right)^\alpha e^{-\left(\frac{\theta}{x}\right)}}{x\Gamma(\alpha)}$
Survival function (for $\alpha \in \mathbb{Z}^+$)	$S(x) = 1 - e^{-\frac{\theta}{x}}(1 + \cdots + \frac{\left(\frac{\theta}{x}\right)^{\alpha-1}}{(\alpha-1)!})$
Mean	$\mu = \frac{\theta}{\alpha-1}$ (if $\alpha > 1$)
Raw moments	$\mu_n' = \theta^n\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}$ (if $\alpha > n$)
Variance	$\mu_2 = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$

Inverse Weibull

Inverse Transformed Gamma with $\alpha = 1$. Inverse of Weibull distribution with $\theta = \frac{1}{\theta}$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^\tau e^{-\left(\frac{\theta}{x}\right)^\tau}}{x}$
Survival function	$1 - e^{-\left(\frac{\theta}{x}\right)^\tau}$
Mean	$\mu = \theta\Gamma\left(1 - \frac{1}{\tau}\right)$ (if $\tau > 1$)
Raw moments	$\mu_n' = \theta^n\Gamma\left(1 - \frac{n}{\tau}\right)$ (if $\tau > n$)
Moment Generating Function	Undefined

Inverse Exponential

Inverse Transformed Gamma with $\tau = \alpha = 1$, inverse of exponential with $\theta = \frac{1}{\theta}$.

Support	$[0, \infty)$
Density function	$f(x) = \frac{\theta e^{-\left(\frac{\theta}{x}\right)}}{x^2}$
Survival function	$1 - e^{-\frac{\theta}{x}}$
Mean	Undefined

Linear Exponential Family

Density	$f_\theta(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$
mean	$\mu(\theta) = \frac{q'(\theta)}{q(\theta)r'(\theta)}$
Variance	$\mu_2(\theta) = \frac{\mu'(\theta)}{r'(\theta)}$

Normal

Support	$(-\infty, \infty)$
Density function	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Mean	$\mu = \mu$
Variance	σ^2
Moment Generating Function	$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Beta

Support	$[0, 1]$
Density function	$f(x) = x^{\alpha-1}(1-x)^{\beta-1}$
Mean	$\mu = \frac{\alpha}{\alpha+\beta}$
Variance	$\frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$
Function	

Uniform

Scaled Beta with $\alpha = \beta = 1$.

Support	$[a, b]$
Density function	$f(x) = \frac{1}{b-a}$ (for $a < x < b$)
Survival funtion	$S(x) = \frac{b-x}{b-a}$ (for $a \leqslant x \leqslant b$)
Mean	$\mu = \frac{a+b}{2}$
Variance	$\frac{(b-a)^2}{12}$
Moment Generating Function	$M(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

Log-Normal

Exponential of a normal distribution.

Support	$[0, \infty)$
Density function	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}}$
Mean	$e^{\mu + \frac{\sigma^2}{2}}$
Variance	$e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$
Moment Generating Function	undefined

Continuous Distributions: Extreme Value Distributions

General Extreme Value Distribution

Distibution function $H_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-\frac{1}{\xi}}} & \text{if } \xi \neq 0 \\ e^{-e^{-x}} & \text{otherwise} \end{cases}$

Gumbel Distribution

Sometimes add scale parameter θ and location parameter μ.

Support	$(-\infty, \infty)$
Distibution function	$F(x) = e^{-e^{-x}}$
Density function	$f(x) = e^{-x}e^{-e^{-x}}$
Mean	0.57721566
Variance	$\frac{\pi^2}{6}$
Moment Generating Function	$M(t) = \Gamma(1-t)$

Fréchet Distribution

Sometimes add location parameter. This is an inverse Weibull distribution.

Support	$[0, \infty)$
Distibution function	$F(x) = e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}$ $(x \geqslant 0)$
Density function	$f(x) = \alpha x^{-\alpha-1}\theta^{\alpha} e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}$ $(x \geqslant 0)$
Mean	$\theta\Gamma\left(1 - \frac{1}{\alpha}\right)$ $(\alpha > 1)$
$\mathbb{E}(X^k)$	$\theta^k\Gamma\left(1 - \frac{k}{\alpha}\right)$ $(k < \alpha)$
Moment Generating	Undefined Function

Weibull EV Distribution

Sometimes add location parameter.

Support	$(-\infty, 0]$
Distibution function	$F(x) = e^{-\left(\frac{x}{\theta}\right)^{\alpha}}$
Density function	$f(x) = \alpha x^{\alpha-1}\theta^{-\alpha} e^{-\left(\frac{x}{\theta}\right)^{\alpha}}$ $(x \leqslant 0)$
Mean	$-\theta\Gamma\left(1 + \frac{1}{\alpha}\right)$
Variance	$\theta^2\left(\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2\right)$

Continuous Distributions: Generalised Pareto Distribution

Support	$[0,\infty)$ if $\xi \geqslant 0$, $\left[0,-\frac{\beta}{\xi}\right]$ if $\xi < 0$
Survival function	$S(x) = \begin{cases} \left(1+\xi\frac{x}{\beta}\right)^{-\frac{1}{\xi}} & \xi \neq 0 \\ e^{-\frac{x}{\beta}} & \xi = 0 \end{cases}$
density function	$f(x) = \begin{cases} \frac{1}{\beta}\left(1+\xi\frac{x}{\beta}\right)^{-\frac{\xi+1}{\xi}} & \xi \neq 0 \\ \frac{1}{\beta}e^{-\frac{x}{\beta}} & \xi = 0 \end{cases}$

- For $\xi > 0$ this is the Pareto distribution.
- For $\xi = 0$ this is the exponential distribution.
- For $\xi < 0$ this is a scaled β distribution with $\alpha = 1$.

Hill estimator

$$\hat{\alpha}_j^H = \left(\sum_{k=j+1}^{n} \frac{\log(x_{(k)}) - \log(x_{(j)})}{n-j+1}\right)^{-1}$$

$$\hat{S}^H(x) = \frac{j}{n}\left(\frac{x}{x_{(n-j)}}\right)^{-\hat{\alpha}_j^H}$$

Discrete Distributions

Binomial

Probability	$p_k = \binom{n}{k}p^k(1-p)^{n-k}$
mean	$\mu = np$
raw moments	$\mathbb{E}(X\cdots(X+1-m)) = n\cdots(n+1-m)p^m$
Variance	$\mu_2 = np(1-p)$
p.g.f.	$P(z) = (1-p+pz)^n$
$(a,b,0)$-class	$a = -\frac{p}{1-p}$, $b = \frac{(n+1)p}{1-p}$
zero-truncated probability	$p_1^T = \frac{np(1-p)^{n-1}}{1-(1-p)^n}$

Poisson

Limit of binomial as $n \to \infty$, $p \to 0$ with $np = \lambda$.

Probability	$p_k = e^{-\lambda}\frac{\lambda^k}{k!}$
mean	$\mu = \lambda$
raw moments	$\mathbb{E}(X(X-1)\cdots(X+1-m)) = \lambda^m$
Variance	$\mu_k = \lambda$
p.g.f.	$P(z) = e^{\lambda(z-1)}$
$(a,b,0)$-class	$a = 0$, $b = \lambda$
zero-truncated probability	$p_1^T = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$

Negative Binomial

- Gamma mixture of Poisson distributions where λ follows a gamma distribution with $\theta = \beta$ and $\alpha = r$.
- Number of successes before r failures if probability of success is $\frac{\beta}{1+\beta}$.
- Compound Poisson-Logarithmic distribution, where $\lambda = r\log\left(\frac{1}{1+\beta}\right)$ and $a = \frac{\beta}{1+\beta}$.

Probability	$p_k = \binom{k+r-1}{k}\left(\frac{\beta}{1+\beta}\right)^k\left(\frac{1}{1+\beta}\right)^r$
	$=\frac{r(r+1)\cdots(r+k-1)}{k!}\left(\frac{\beta}{1+\beta}\right)^k\left(\frac{1}{1+\beta}\right)^r$
mean	$\mu = r\beta$
Variance	$\mu_n = r\beta(1+\beta)\cdots(n-1+\beta)$
p.g.f.	$P(z) = \left(\frac{1}{1+\beta-\beta z}\right)^r$
$(a,b,0)$-class	$a = \frac{\beta}{1+\beta}$, $b = \frac{(r-1)\beta}{1+\beta}$
zero-truncated probability	$p_1^T = \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}$

$(a,b,0)$ and $(a,b,1)$ Classes

$p_k = \left(a+\frac{b}{k}\right)p_{k-1}$ for $k > 1$ (and for $k > 0$ in the $(a,b,0)$ class).

mean	$\mu = \frac{a+b}{1-a}$
Variance	$\mu_2 = \frac{a+b}{(1-a)^2}$
p.g.f.	$P(z) = \left(\frac{1-az}{1-a}\right)^{-\left(1+\frac{b}{a}\right)}$
zero-truncated mean	$\mu = \frac{a+b}{(1-a)\left(1-(a+b)^{1+\frac{b}{a}}\right)}$
zero-truncated probability	$p_1^T = \frac{a+b}{(1-a)^{-\left(1+\frac{b}{a}\right)}-1}$

Logarithmic distribution

Negative binomial with $r = 0$. $(a,b,1)$-class with $a+b = 0$, $a = \frac{\beta}{1+\beta}$.

zero-truncated probability	$p_1^T = \frac{-a}{\log(1-a)} = \frac{\beta}{(1+\beta)\log(1+\beta)}$
probability	$p_n = \frac{a^{n-1}}{n} p_1$
mean	$\mu = \frac{-a}{(1-a)(\log(1-a))} = \frac{\beta}{\log(1+\beta)}$
Variance	$\mu_2 = \frac{p_1 - p_1^2}{(1-a)^2} = \frac{\beta(1+\beta)}{\log(1+\beta)} - \frac{\beta}{\log(1+\beta)^2}$
p.g.f.	$P(z) = -\frac{\log(1-az)}{\log(1-a)}$
$(a,b,1)$-class	$a = \frac{\beta}{1+\beta}$, $b = -\frac{\beta}{1+\beta}$

Compound Distributions

Moments:

Let the moments of the primary distribution be $\mu, \mu_2, \mu_3, \ldots$, and the moments of the secondary distribution by $\nu, \nu_2, \nu_3, \ldots$. The moments of the compound distribution are given by:

$\mu\nu$
$\mu\nu_2 + \mu_2\nu^2$
$\mu\nu_3 + \mu_2\nu\nu_2 + \mu_3\nu^3$.

Recursive formula:

If the primary distribution is a member of the $(a,b,1)$-class, the probability mass function is defined as

$$f_S(k) = \frac{(p_1 - (a+b)p_0) f_X(k) + \sum_{i=1}^{k} \left(a + \frac{bi}{k}\right) f_X(i) f_S(k-i)}{1 - a f_X(0)}$$

where:

- f_X is the probability mass function of the secondary distribution
- f_S is the probability mass function of the compound distribution
- p_n is the probability that the primary distribution is n (so $p_n = \left(a + \frac{b}{n}\right) p_{n-1}$)

Information Criteria

- Akaike information criterion (AIC) $l(\theta; x) - p$
- Schwartz Bayesian criterion/Bayes information criterion (BIC) $l(\theta; x) - \frac{p\log(n)}{2}$

Hypothesis Tests

Anderson-Darling test

- Test statistic $n \int_t^u \frac{(F_n(x) - F^*(x))^2}{F^*(x)(1-F^*(x))} f^*(x)\,dx$
- For complete data, given by the formula:

$$-nF^*(u) + n\sum_{i=1}^{k}(F_n(y_i))^2\big(\log(F^*(y_{i+1})) - \log(F^*(y_i))\big)$$
$$+n\sum_{i=0}^{k}(1 - F_n(y_i))^2\big(\log(1 - F^*(y_i)) - \log(1 - F^*(y_{i+1}))\big)$$

where

- n is sample size.
- Unique observed values are $t = y_0 < y_1 < \cdots < y_k < y_{k+1} = u$
- t is the (left) truncation point (can be $-\infty$ or 0 if no truncation).
- u is the (right) censorship point (can be ∞ if no censorship).

Claims Reserving

Bühlmann-Straub Credibility Reserves

$$\hat{v} = \frac{1}{I}\sum_{i=0}^{I-1}\frac{1}{I-i}\sum_{j=0}^{I-i}\hat{\gamma}_j\left(\frac{X_{ij}}{\hat{\gamma}_j} - \widehat{C_{i,J}}\right)^2$$

$$\hat{a} = \frac{\sum_{i=0}^{I}\hat{\beta}_{I-i}\left(\widehat{C_{i,J}} - \overline{C}\right)^2 - I\hat{v}}{\sum_{i=0}^{I}\hat{\beta}_{I-i} - \frac{1}{\sum_{i=0}^{I}\hat{\beta}_{I-i}}\sum_{i=0}^{I}\hat{\beta}_{I-i}^2}$$

where

$$\overline{C} = \frac{\sum_{i=0}^{I} C_{i,I-i}}{\sum_{i=0}^{I}\hat{\beta}_{I-i}}$$

Then estimate

$$
\begin{aligned}
Z_i &= \frac{\hat{\beta}_{I-i}}{\hat{\beta}_{I-i} + \frac{\hat{v}}{\hat{a}}} \\
\hat{\mu} &= \frac{\sum_{i=0}^{I} Z_i \hat{C}_{i,J}}{\sum_{i=0}^{I} Z_i} \\
\hat{C}_{i,J}^{\mathrm{BS}} &= Z_i \hat{C}_{i,J} + (1 - Z_i)\hat{\mu} \\
\hat{C}_{i,J}^{\mathrm{BS2}} &= C_{i,I-i} + (1 - \hat{\beta}_j)\hat{C}_{i,J}^{\mathrm{BS}}
\end{aligned}
$$

Mack's Model

$$\hat{\sigma}_j^2 = \frac{1}{I-1-j} \sum_{i=0}^{I-1-j} C_{ij} \left(f_{ij} - \hat{f}_j \right)^2 \qquad \text{for} j \leqslant I-2$$

Use $\quad \hat{\sigma}_{J-1}^2 = \min\left(\sigma_{J-2}^2, \sigma_{J-3}^2, \frac{\sigma_{J-2}^4}{\sigma_{J-3}^2} \right) \qquad$ when $I = J$

- $$\mathrm{Var}(C_{i,J} | C_{i,I-i}) \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} \hat{C}_{i,j}}$$

- $$\mathbb{E}\left(\left(\hat{C}_{i,J} - \mathbb{E}(C_{i,j}|D_I) \right)^2 \right) \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} S_j}$$

- $$\mathbb{E}\left(\left(\hat{C}_{i,J} - \mathbb{E}(C_{i,j}|D_I) \right) \left(\hat{C}_{i',J} - \mathbb{E}(C_{i',j}|D_I) \right) \right) \approx \hat{C}_{i,J}\hat{C}_{i',J} \sum_{j=I-(i \wedge i')}^{J} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} S_j}$$

  where $S_j = \sum_{i=0}^{I-1-j} C_{i,j}$.