5 Continuous Distributions

5.3 Creating New Distributions

1 The mean of X is $\frac{\alpha}{\alpha+\beta}$, and the variance is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$. Thus, the standardised variable is $\sqrt{\frac{(\alpha+\beta)^2(\alpha+\beta+1)}{\alpha\beta}}\left(X - \frac{\alpha}{\alpha+\beta}\right)$. The density is therefore

$$f_{X^{\mathrm{s}}}(x) = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}} f_X\left(\sqrt{\frac{(\alpha+\beta)^2(\alpha+\beta+1)}{\alpha\beta}} x + \frac{\alpha}{\alpha+\beta}\right)$$

$$= \frac{\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}\left(\sqrt{\frac{(\alpha+\beta)^2(\alpha+\beta+1)}{\alpha\beta}} x + \frac{\alpha}{\alpha+\beta}\right)^{\alpha-1}\left(\frac{\beta}{\alpha+\beta} - \sqrt{\frac{(\alpha+\beta)^2(\alpha+\beta+1)}{\alpha\beta}} x\right)^{\beta-1}}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}$$

2 The gamma distribution has mean $\alpha\theta$ and variance $\alpha\theta^2$. Thus, the standardised gamma distribution is $\frac{X-\alpha\theta}{\sqrt{\alpha}\theta}$. The density is

$$f_{X^{\mathrm{S}}}(x) = \sqrt{\alpha}\theta f_X\left(\sqrt{\alpha}\theta x + \alpha\theta\right) = \sqrt{\alpha}\theta \frac{\left(\sqrt{\alpha}\theta x + \alpha\theta\right)^{\alpha-1} e^{-\frac{\sqrt{\alpha}\theta\left(\sqrt{\alpha}\theta x+\alpha\theta\right)}{\theta}}}{\theta^\alpha \gamma(\alpha)} = \frac{\sqrt{\alpha}\left(\sqrt{\alpha}x+\alpha\right)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\left(x+\sqrt{\alpha}\right)}$$

The density of the square of the standardised X^{S} is therefore given by

$$\begin{aligned}
f_{X^{\mathrm{S}^2}}(x) &= \begin{cases} \frac{x}{2}\left(f_{X^{\mathrm{S}}}(\sqrt{x}) + f_{X^{\mathrm{S}}}(-\sqrt{x})\right) & \text{if } x < \alpha \\ \frac{x}{2} f_{X^{\mathrm{S}}}(\sqrt{x}) & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{x}{2}\left(\frac{\sqrt{\alpha}\left(\sqrt{\alpha x}+\alpha\right)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\left(\sqrt{x}+\sqrt{\alpha}\right)} + \frac{\sqrt{\alpha}\left(\alpha-\sqrt{\alpha x}\right)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\left(\sqrt{\alpha}-\sqrt{x}\right)}\right) & \text{if } x < \alpha \\ \frac{x}{2}\frac{\sqrt{\alpha}\left(\sqrt{\alpha x}+\alpha\right)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\left(\sqrt{x}+\sqrt{\alpha}\right)} & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{\sqrt{\alpha}x}{2\Gamma(\alpha)} e^{-\alpha^{\frac{3}{2}}}\left(\left(\sqrt{\alpha x}+\alpha\right)^{\alpha-1} e^{-\alpha\sqrt{x}} + \left(\alpha - \sqrt{\alpha x}\right)^{\alpha-1} e^{\alpha\sqrt{x}}\right) & \text{if } x < \alpha \\ \frac{\sqrt{\alpha}x\left(\sqrt{\alpha x}+\alpha\right)^{\alpha-1}}{2\Gamma(\alpha)} e^{-\alpha\left(\sqrt{x}+\sqrt{\alpha}\right)} & \text{otherwise} \end{cases}
\end{aligned}$$

3 Let X_i be the percentage increase on day i. Let $Y_i = \log\left(1 + \frac{X_i}{100}\right)$. Let V_i be the value of the investment after day i. We have that $V_i = V_{i-1}(1+X_i)$, so $\log(V_i) = \log(V_{i-1}) + Y_i$. Thus, $\log(V_{365}) = \log(V_0) + \sum_{i=1}^{365} Y_i$. By the CLT, we can approximate $\frac{1}{365}\sum_{i=1}^{365} Y_i$ by a normal distribution with mean $\mathbb{E}(Y_i)$ and variance $\frac{\mathrm{Var}(Y_i)}{365}$. We can use the Taylor series approximation $Y_i \approx \frac{X_i}{100}$, so that $\mathbb{E}(Y_i) = \frac{\mathbb{E}(X_i)}{100} = 0.0004$ and $\mathrm{Var}(Y_i) = \frac{\mathrm{Var}(X_i)}{100^2} = 0.0005$. Thus $\sum_{i=1}^{365} Y_i$ is approximately normally distributed with mean $365 \times 0.0004 = 0.146$ and variance $365 \times 0.0005 = 0.1825$. Thus, the distribution of V_{365} is approximately log-normal with $\mu = \log(V_0) + 0.146$ and $\sigma^2 = 0.1825$. [The mean of this is $e^{\mu+\frac{\sigma^2}{2}} = V_0 e^{0.146+\frac{0.1825}{2}} = 1.26775801767V_0$, and the variance is $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = V_0^2\left(e^{0.292+0.365} - e^{0.292+0.1825}\right) = 0.3217862639V_0^2$.]

4 Let the losses be $10000X_1$ and $10000X_2$. We have that

$$\begin{aligned}
P(X_1+X_2>5) &= \mathbb{E}_{X_1}(P(X_2>5-X_1)) \\
&= \mathbb{E}_{X_1}\left(\left\{\begin{array}{ll}\left(\frac{1}{1+5-X_1}\right)^4 & \text{if } X_1<5 \\ 1 & \text{otherwise}\end{array}\right.\right) \\
&= P(X_1>5)+\int_0^5 \frac{4}{(x+1)^5}\left(\frac{1}{6-x}\right)^4\,dx \\
&= \left(\frac{1}{1+5}\right)^4+4\int_0^5 (x+1)^{-5}(6-x)^{-4}\,dx \\
&= 6^{-4}+4\int_0^5\left(\frac{1}{7^4(x+1)^5}+\frac{4}{7^5(x+1)^4}+\frac{10}{7^6(x+1)^3}+\frac{20}{7^7(x+1)^2}+\frac{35}{7^8(x+1)}+\frac{35}{7^8(6-x)}\right. \\
&\qquad\left.+\frac{15}{7^7(6-x)^2}+\frac{5}{7^6(6-x)^3}+\frac{1}{7^5(6-x)^4}\right)dx \\
&= 6^{-4}+\frac{4}{4\times 7^4}\left(1-6^{-4}\right)+\frac{16}{3\times 7^5}\left(1-6^{-3}\right)+\frac{40}{2\times 7^6}\left(1-6^{-2}\right)+\frac{80}{1\times 7^7}\left(1-6^{-1}\right)+\frac{140}{7^8}\log(6) \\
&\qquad+\frac{140}{7^8}\log(6)+\frac{60}{1\times 7^7}\left(1-6^{-1}\right)+\frac{20}{2\times 7^6}\left(1-6^{-2}\right)+\frac{4}{3\times 7^5}\left(1-6^{-3}\right) \\
&= 0.00200997262988
\end{aligned}$$

5

(a) We have

$$P(X > 10000) = \mathbb{E}\left(P(X > 10000|\Theta)\right) = \mathbb{E}\left(\left(\frac{\Theta}{\Theta + 10000}\right)^{2.5}\right) = \int_0^\infty \frac{\theta^3 e^{-\frac{\theta}{800}}}{6 \times 800^4}\left(\frac{\theta^{2.5}}{(\theta + 10000)^{2.5}}\right) d\theta$$

Numerically we can evaluate this.

```
theta<-seq_len(1000000)/10
sum(exp(-theta/800)*theta^5.5/(theta+10000)^2.5)/(10*6*800^4)
```

This gives $P(X > 10000) = 0.03278301$. [Varying the step size and upper bounds of the numerical integral does not change the result, so this should be sufficiently accurate.]

(b) Given Θ, the expected aggregate loss is $\frac{\Theta}{2.5-1} = \frac{\Theta}{1.5}$, so the expected aggregate loss for a random individual is $\mathbb{E}\left(\frac{\Theta}{1.5}\right) = \frac{4\times 800}{1.5} = \$2,133.33$.

(c) The VaR is the solution to $P(X > a) = 0.01$, which based on (a) becomes

$$\int_0^\infty \frac{\theta^3 e^{-\frac{\theta}{800}}}{6 \times 800^4}\left(\frac{\theta^{2.5}}{(\theta + a)^{2.5}}\right) d\theta = 0.01$$

Trying a few values gives $\mathrm{VaR}(X) = 18823$. The TVaR is then given by

$$\mathrm{TVaR}(X) = 18823 + \frac{\mathbb{E}((X - 18823)_+)}{0.01} = 18823 + 100\mathbb{E}\left(\mathbb{E}((X - 18823)_+|\Theta)\right)$$

Given $\Theta = \theta$, we have

$$\mathbb{E}((X - 18823)_+) = \int_{18823}^\infty \left(\frac{\theta}{\theta + x}\right)^{2.5} dx = \int_{18823+\theta}^\infty \theta^{2.5} u^{-2.5}\, du = \frac{\theta^{2.5}}{1.5(18823 + \theta)^{1.5}}$$

Thus the TVaR is

$$\mathbb{E}\left(\frac{\Theta^{2.5}}{1.5(18823 + \Theta)^{1.5}}\right) = \int_0^\infty \frac{\theta^3 e^{-\frac{\theta}{800}}}{6 \times 800^4}\left(\frac{\theta^{2.5}}{1.5(\theta + 18823)^{1.5}}\right) d\theta$$

Numerically, we can evaluate this

```
theta<-seq_len(1000000)/10
sum(exp(-theta/800)*theta^5.5/(theta+10000)^1.5)/(1.5*10*6*800^4)
```

This gives

$$\mathbb{E}\left(\frac{\Theta^{2.5}}{1.5(18823 + \Theta)^{1.5}}\right) = 157.5151$$

so the TVaR is $18823 + 100 \times 157.5151 = \$34,574.51$.

6 For the normal distribution, the density at $x = 10,000$ is $\frac{1}{3000\sqrt{2\pi}}e^{-\frac{(10000-4000)^2}{2\times 3000^2}} = \frac{e^{-2}}{3000\sqrt{2\pi}} = 0.0000179969888378$. The normal distribution assigns probability $\Phi(2) = 0.9772499$ to $X < 10,000$, so for the probability to be 0.92, the normal density needs to be rescaled by a factor $\frac{0.92}{0.9772499} = 0.941417338595$. Thus the rescaled Pareto density needs to be $0.0000179969888378 \times 0.941417338595 = 0.0000169426773344$.

With parameter θ, the Pareto distribution has probability $\left(\frac{\theta}{\theta+10000}\right)^3$ of exceeding 10000, so it needs to be scaled by a factor of $0.08\left(\frac{\theta+10000}{\theta}\right)^3$ to get the correct probability. After rescaling, we want the Pareto density at 10000, which is given by $f(10000) = \frac{3\theta^3}{(\theta+10000)^4}$ to be 0.0000169426773344. Thus, we need to solve

$$\begin{aligned}
0.08\left(\frac{\theta+10000}{\theta}\right)^3 \frac{3\theta^3}{(\theta+10000)^4} &= 0.0000169426773344 \\
\frac{3}{\theta+10000} &= 0.00021178346668 \\
\theta + 10000 &= 14165.4117152 \\
\theta &= 4165.4117152
\end{aligned}$$

The probability that aggregate claims exceed \$25,000 is therefore

$$0.08\left(\frac{14165.4117152}{4165.4117152}\right)^3 \frac{4165.4117152^3}{(4165.4117152+25000)^3} = 0.08\left(\frac{14165.4117152}{29165.4117152}\right)^3 = 0.0091658608024$$

IRLRPCI 5 Intermediate topics

5.1 Individual risk rating plans

7

For the \$50,000 to \$100,000 increase, we consider only the policies with limit at least \$100,000. For these policies, the total losses limited to \$100,000 are $41,000 + 26000 + 12300 = 79300$, while total losses limited to \$50,000 for the same policies are $34000 + 23000 + 11000 = 68000$. The ILF is therefore $\frac{79300}{68000} = 1.166176$

For the \$100,000 to \$500,000 increase, the ILF is $\frac{31000+13400}{26000+12300} = \frac{44400}{38300} = 1.159269$. The other ILFs are given in the following table

Old Policy limit	New Policy Limit		
	100,000	500,000	1,000,000
50,000	1.166176	1.305882	1.545455
100,000		1.159269	1.382114
500,000			1.268657

If instead, we use incremental factors, the incremental factors are on the diagonal of the above table, and other factors are obtained by multiplying the incremental factors below them, so for example, the ILF from \$50,000 to \$500,000 would be $1.166176 \times 1.159269 = 1.351912$, while the ILF for an increase from \$50,000 to \$1,000,000 is $1.166176 \times 1.159269 \times 1.268657 = 1.715112$. The ILFs based on this incremental method are given in the following table:

Old Policy limit	New Policy Limit		
	100,000	500,000	1,000,000
50,000	1.166176	1.351912	1.715112
100,000		1.159269	1.470715
500,000			1.268657

8

The expected payments per claim are $a\theta - \mathbb{E}((X - 5a\theta)_+)$. By the memoryless property of the exponential distribution $\mathbb{E}((X-5a\theta)_+) = a\theta P(X > 5a\theta) = a\theta e^{-\frac{5a\theta}{a\theta}} = a\theta e^{-5}$. The expected payment per claim is therefore $(1 - e^{-5})a\theta$.

After inflation, the new mean loss per claim is $1.1a\theta$. The expected payment per claim is $1.1a\theta - \mathbb{E}((X - 5a\theta)_+)$, and $\mathbb{E}((X - 5a\theta)_+) = 1.1a\theta e^{-\frac{5a\theta}{1.1a\theta}} = 1.1e^{-\frac{5}{1.1}}a\theta$. The new expected claim payment is therefore $1.1a\theta(1 - e^{-\frac{5}{1.1}})$. The percentage increase in claim payments is therefore $\frac{1.1(1-e^{-\frac{5}{1.1}})}{1-e^{-5}} - 1 = 9.5706\%$.

9 We will use a normal approximation to aggregate losses.
(a) The expected value of a Pareto distribution censored at \$50,000 is

$$10000\int_0^5 \frac{1}{(1+u)^3}\,du = 10000\int_1^6 a^{-3}\,da = 10000\left[-\frac{a^{-2}}{2}\right]_1^6 = \frac{10000}{2}\left(1-\frac{1}{6^2}\right) = \$4,861.11$$

The expected square of the payment censored at \$50,000 is

$$10000^2\int_0^5 \frac{2u}{(1+u)^3}\,du = 10000^2\int_1^6 2(a-1)a^{-3}\,da = 10000^2\left[a^{-2}-2a^{-1}\right]_1^6 = 10000^2\left(1+\frac{1}{6^2}-\frac{2}{6}\right) = 69,444,444$$

so the variance is $69444444 - 4861.11^2 = 45,814,043$.

The mean aggregate loss is therefore $100 \times 4861.11 = \$486,111.11$ and the variance of aggregate loss is $100 \times 45814043 + 100 \times 4861.11^2 = 6,944,444,444$. Using a normal approximation, the 95th percentile of aggregate losses is $486,111.11 + 1.645\sqrt{6,944,444,444} = \$623182.25$. The risk loading as a percentage of the gross rate is therefore $\frac{623182.25-486111.11}{623182.25} = 21.99535\%$.

(b) Censored at \$100,000, the expected payment per loss is

$$10000\int_0^{10} \frac{1}{(1+u)^3}\,du = 10000\int_1^{11} a^{-3}\,da = 10000\left[-\frac{a^{-2}}{2}\right]_1^{11} = \frac{10000}{2}\left(1-\frac{1}{11^2}\right) = \$4,958.69$$

The expected squared loss is

$$10000^2\int_0^{10} \frac{2u}{(1+u)^3}\,du = 10000^2\int_1^{11} 2(a-1)a^{-3}\,da = 10000^2\left[a^{-2}-2a^{-1}\right]_1^{11} = 10000^2\left(1+\frac{1}{11^2}-\frac{2}{11}\right) = 82,644,628$$

and the variance is $82644628 - 4958.69^2 = 58056144$.

The mean aggregate loss is $100 \times 4958.68 = \$495,867.77$ and the variance is $100 \times 82644628 = 8264462810$. The 95th percentile is then $495,867.77 + 1.645\sqrt{8264462810} = \$645399.92$. The risk loading as a percentage of the gross rate is therefore $\frac{645399.92-495867.77}{645399.92} = 23.16891\%$.

10

The total number of claims is 3365. We calculate the average loss for each policy limit. For example for policy limit \$10,000, the total claims would be $6850000 + 1065 \times 10000 = \$17,500,000$, so the average claim would be $\frac{17500000}{3365} = \$5,200.59$.

Policy limit	Total claimed	Average per claim	Risk Charge	Total charge
100,000	36,950,000	10980.68	2411.51	\$13,392.19
500,000	52,350,000	15557.21	4840.53	\$20,397.74
1,000,000	58,450,000	17369.99	6034.33	\$23,404.31

The ILF from \$100,000 to \$500,000 is $\frac{20397.74}{13392.19} = 1.523107$. and to \$1,000,000, it is $\frac{23404.31}{13392.19} = 1.747609$.
[For the pure premium, the ILFs would be 1.416779 and 1.581867 respectively.]

11 Let the expected payment without the deductible be a, and the expected payment with the deductible be b. The expected payment on a policy with limit \$1,000 is $a - b$. Thus from the definition of the ILF, we have $4.62(a - b) = a$. The loss elimination ratio is $1 - \frac{b}{a} = \frac{a-b}{a} = \frac{1}{4.62} = 21.645\%$.

12 Let a_{2021} and a_{2022} be the expected losses with limit \$1,000,000 in 2021 and 2022 respectively. Let b_{2021} and b_{2022} be the expected losses with limit \$2,000,000 in 2021 and 2022 respectively. In 2021 the premium for a policy with limit \$1,000,000 is $1.25a_{2021}$. Thus, the premium for a policy with limit \$2,000,000 is $1.36 \times 1.25a_{2021} = 1.7a_{2021}$. Hence the premium for the reinsurance is $(1.7 - 1.25)a_{2021} = 0.45a_{2021}$. If the loading is l, then we have $(1 + l)(b_{2021} - a_{2021}) = 0.45a_{2021}$.

Similarly, the premium for reinsurance in 2022 is $(1.35 \times 1.25 - 1.25)a_{2022} = 0.4375a_{2022}$. From the trend factor, we have $a_{2022} = 1.052a_{2021}$, so the premium in 2022 is $0.4375 \times 1.052a_{2021} = 0.46025a_{2021}$. Thus $(1 + l)(b_{2022} - 1.052a_{2021}) = 0.46025a_{2021}$. Substituting $b_{2022} = 1.044b_{2021}$ gives

$$(1 + l)(1.044b_{2021} - 1.052a_{2021}) = 0.46025a_{2021}$$

Subtracting $1.044(1 + l)(b_{2021} - a_{2021}) = 1.044 \times 0.45a_{2021}$ from both sides gives

$$(1 + l)(-0.008a_{2021}) = (0.46025 - 1.044 \times 0.45)a_{2021} = -0.00955a_{2021}$$

$$1 + l = \frac{0.00955}{0.008} = 1.19375$$

so the loading is 19.38%.

SN2 Extreme Value Theory

General Extreme Value Distributions

13

(a) We can use the following R code:

```
set.seed(12345)  # For reproducability
x<-rnorm(10000000) # Simulate normal random variables.
bmaxsamps<-list() # A list of block maxima.
snum<-list()  # The number of blocks.

for(n in seq_len(100)){
    ssize<-floor(10000/n) # calculate number of blocks
    snum[[n]]<-rep(n*1000,ssize)
    M<-x[seq_len(n*1000*ssize)] # discard excess samples
    dim(M)<-c(n*1000,ssize) # Make M into a matrix
    bmaxsamps[[n]]<-apply(M,2,max) # Take maximum of each column
}

all_samples<-data.frame(size=unlist(snum),max=unlist(bmaxsamps))
### The first column of this data frame is the sample size
### The second column is the block maxima.

library(ggplot2)

ggplot(data=all_samples,mapping=aes(x=size,y=max))+
    geom_violin(mapping=aes(group=as.factor(size)))
```

(b) We first plot some graphs:

```
### Use GAM to fit a smooth curve.

ggplot(data=all_samples,mapping=aes(x=size,y=max))+
    geom_violin(mapping=aes(group=as.factor(size)))+
    geom_smooth(method="gam")


### Try log-transforming block size
ggplot(data=all_samples,mapping=aes(x=log(size),y=max))+
    geom_violin(mapping=aes(group=as.factor(size)))+
    geom_smooth(method="gam")

### Now the pattern is linear. Fit a linear model

lm(max~log(size),data=all_samples)
```

```
### calculate standard deviations of block maxima for each sample size

block_sd<-data.frame(size=seq_len(100)*1000,sd=rep(NA,100))
for(n in seq_len(100)){
    block_sd$sd[n]<-sqrt(mean((all_samples$max[all_samples$size==n*1000]-0.2581*log(n*1000))
}

### plot them
ggplot(block_sd,mapping=aes(x=size,y=sd))+geom_point()+geom_smooth()

### After a bit of experimentation, we find a function which fits reasonably.
ggplot(block_sd,mapping=aes(x=size,y=sd))+geom_point()+geom_smooth()+
    geom_line(mapping=aes(y=0.285+400/(size+5000)))
```

The plot suggests a log-linear model, and using linear regression give $\mathbb{E}(\text{max}) = 1.4736 + 0.2581 \log(\text{size})$.

To fit location, we calculate the standard deviation of residuals of this fitting and then fit a function to them.

(c) We get the following plot of the rescaled block maxima.

```
ggplot(data=all_samples,mapping=aes(x=size,
                                     y=(max-0.2581*log(size))/(0.285+400/(size+5000)),
                                     group=as.factor(size)))+
    geom_violin()
```

14

(a) We can use the following R code:

```
set.seed(12345)  # For reproducability
x<-rexp(10000000) # Simulate normal random variables.
bmaxsampsexp<-list() # A list of block maxima.
snumexp<-list()  # The number of blocks.

for(n in seq_len(100)){
     ssize<-floor(10000/n) # calculate number of blocks
     snumexp[[n]]<-rep(n*1000,ssize)
     M<-x[seq_len(n*1000*ssize)] # discard excess samples
     dim(M)<-c(n*1000,ssize) # Make M into a matrix
     bmaxsampsexp[[n]]<-apply(M,2,max) # Take maximum of each column
}

all_samples_exp<-data.frame(size=unlist(snumexp),max=unlist(bmaxsampsexp))
### The first column of this data frame is the sample size
### The second column is the block maxima.

library(ggplot2)

ggplot(data=all_samples_exp,mapping=aes(x=size,y=max))+
     geom_violin(mapping=aes(group=as.factor(size)))
```

(b) We first plot some graphs:

```
### Use GAM to fit a smooth curve.

ggplot(data=all_samples_exp,mapping=aes(x=size,y=max))+
     geom_violin(mapping=aes(group=as.factor(size)))+
     geom_smooth(method="gam")


### Try log-transforming block size
ggplot(data=all_samples_exp,mapping=aes(x=log(size),y=max))+
     geom_violin(mapping=aes(group=as.factor(size)))+
     geom_smooth(method="gam")

### Now the pattern is linear. Fit a linear model

lm(max~log(size),data=all_samples_exp)

### Coefficient seems to be 1
### Can actually show that it is 1.

mean(all_samples_exp$max-log(all_samples_exp$size))
```

```
### 0.6220843


### calculate standard deviations of block maxima for each sample size

 block_sd_exp<-data.frame(size=seq_len(100)*1000,sd=rep(NA,100))
 for(n in seq_len(100)){
     block_sd_exp$sd[n]<-sqrt(mean((all_samples_exp$max[all_samples_exp$size==n*1000]-log(n*1
 }

### plot them
ggplot(block_sd,mapping=aes(x=size,y=sd))+geom_point()+geom_smooth()
### close to linear, but that doesn't make any sense in the long run.


 ggplot(block_sd,mapping=aes(x=size,y=log(sd)))+geom_point()+geom_smooth(method="gam")
### Again, linear seems a reasonable fit.

 lm(log(sd)~size,data=block_sd_exp)

### fits sd=e^{0.2668-7.018e-07*size}
```

The plot suggests a log-linear model, and using linear regression give $\mathbb{E}(\text{max}) = 1.4736 + 0.2581 \log(\text{size})$.

To fit location, we calculate the standard deviation of residuals of this fitting and then fit a function to them.

(c) We get the following plot of the rescaled block maxima.

```
ggplot(data=all_samples_exp,mapping=aes(x=size,
                                        y=(max-log(size))/(exp(0.2668-7.018e-07*size),
                                        group=as.factor(size)))+
    geom_violin()
```

In fact, we can analytically solve this $F_n(x) = F(x)^n = \left(1 - e^{-x}\right)^n$. In particular, if $x = \log(n) + c$, then we have $F_n(x) = \left(1 - e^{-\log(n)-c}\right)^n = \left(1 - \frac{e^{-c}}{n}\right)^n \to e^{e^{-c}}$ as $n \to \infty$. Thus, we see that the distribution of $M_n - \log(n)$ converges to a Gumbel distribution.

15

Lemma 1. *For $x \geqslant e$,*

$$\Phi(-x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} + O\left(\frac{e^{-\frac{x^2}{2}}\log(x)}{\sqrt{2\pi}x^3}\right)$$

Proof.

$$\begin{aligned}
\Phi(-x) &= \int_{-\infty}^{-x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy \\
&= \int_{-\infty}^{-x} ye^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}y}\,dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} ye^{-\frac{y^2}{2}} \left(\frac{1}{x} - \left(\frac{1}{x} - \frac{1}{y}\right)\right) dy \\
&= \frac{1}{\sqrt{2\pi}x} \int_{-\infty}^{-x} ye^{-\frac{y^2}{2}}\,dy + \int_{-\infty}^{-x} ye^{-\frac{y^2}{2}} \left(\left(\frac{1}{x} + \frac{1}{y}\right) - \frac{1}{x}\right) dy \\
&= \frac{-1}{\sqrt{2\pi}x}\left[-e^{-\frac{y^2}{2}}\right]_{-\infty}^{-x} + \int_{-\infty}^{-x} \left(\frac{1}{y} + \frac{1}{x}\right) \frac{ye^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy \\
&= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} + \int_{-\infty}^{-x} \left(\frac{1}{y} + \frac{1}{x}\right) \frac{ye^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy
\end{aligned}$$

Thus, it is sufficient to show that

$$\left|\int_{-\infty}^{-x} \left(\frac{1}{y} + \frac{1}{x}\right) ye^{-\frac{y^2}{2}}\,dy\right| \leqslant \frac{3e^{-\frac{x^2}{2}}\log(x)}{x^3}$$

We have

$$\begin{aligned}
\left|\int_{-\infty}^{-x} \left(\frac{1}{y} + \frac{1}{x}\right) ye^{-\frac{y^2}{2}}\,dy\right| &= \left|\int_{-\infty}^{-x-\delta} \left(\frac{1}{y} + \frac{1}{x}\right) ye^{-\frac{y^2}{2}}\,dy + \int_{-x-\delta}^{-x} \left(\frac{1}{y} + \frac{1}{x}\right) ye^{-\frac{y^2}{2}}\,dy\right| \\
&\leqslant \left|\int_{-\infty}^{-x-\delta} \left(\frac{1}{x}\right) ye^{-\frac{y^2}{2}}\,dy\right| + \left|\int_{-x-\delta}^{-x} \left(\frac{1}{x} - \frac{1}{x+\delta}\right) ye^{-\frac{y^2}{2}}\,dy\right| \\
&\leqslant \frac{e^{-\frac{(x+\delta)^2}{2}}}{x} + \frac{\delta}{x^2}e^{-\frac{x^2}{2}} \\
&\leqslant \frac{e^{-\frac{x^2}{2}}}{x}\left(e^{-\delta x} + \frac{\delta}{x}\right) \\
&\leqslant \frac{e^{-\frac{x^2}{2}}}{x}\left(e^{-2\log(x)} + \frac{2\log(x)}{x^2}\right) \\
&\leqslant \frac{3\log(x)e^{-\frac{x^2}{2}}}{x^3}
\end{aligned}$$

□

For the log-normal distribution, we have $F(x) = \Phi\left(\frac{\log(x)-\mu}{\sigma}\right)$, so

$$P\left(\frac{M_n - d_n}{c_n} < x\right) = P(M_n < c_n x + d_n) = F\left(c_n x + d_n\right)^n = \Phi\left(\frac{\log\left(c_n x + d_n\right) - \mu}{\sigma}\right)^n$$

This gives

$$\begin{aligned}
\log\left(P\left(\frac{M_n - d_n}{c_n} < x\right)\right) &= n\log\left(\Phi\left(\frac{\log\left(c_n x + d_n\right) - \mu}{\sigma}\right)\right) \\
&= n\log\left(1 - \Phi\left(-\frac{\log\left(c_n x + d_n\right) - \mu}{\sigma}\right)\right) \\
&= -n\Phi\left(-\frac{\log\left(c_n x + d_n\right) - \mu}{\sigma}\right) + O\left(n\Phi\left(-\frac{\log\left(c_n x + d_n\right) - \mu}{\sigma}\right)^2\right) \\
&= -n\Phi\left(-r_n\right) + O\left(n\Phi\left(-r_n\right)^2\right) \\
&= -n\frac{e^{-\frac{r_n^2}{2}}}{\sqrt{2\pi}r_n} + O\left(n\frac{e^{-\frac{r_n^2}{2}}\log(r_n)}{\sqrt{2\pi}r_n^3}\right) + O\left(n\left(\frac{e^{-\frac{r_n^2}{2}}}{\sqrt{2\pi}r_n}\right)^2\right)
\end{aligned}$$

where $r_n = \frac{\log(c_n x + d_n) - \mu}{\sigma} = \frac{\log(d_n) + \log\left(1 + \frac{c_n}{d_n}x\right) - \mu}{\sigma}$. For any x, we see that $r_n \to \infty$ as $n \to \infty$, so for any x,

$$\log\left(P\left(\frac{M_n - d_n}{c_n} < x\right)\right) \to -n\frac{e^{-\frac{r_n^2}{2}}}{\sqrt{2\pi}r_n}$$

Furthermore, $\frac{c_n}{d_n} \to 0$, so we can use $\log\left(1 + \frac{c_n}{d_n}x\right) = \frac{c_n}{d_n}x + O\left(\frac{c_n^2}{d_n^2}x^2\right)$, which gives $r_n = \frac{\log(d_n) + \frac{c_n}{d_n}x + O\left(\frac{c_n^2}{d_n^2}x^2\right) - \mu}{\sigma}$ and so

$$e^{-\frac{r_n^2}{2}} = e^{-\frac{1}{2\sigma^2}\left((\log(d_n)-\mu)^2 + 2\frac{c_n(\log(d_n)-\mu)}{d_n}x + O\left(\frac{c_n^2(\log(d_n)-\mu)}{d_n}x^2\right)\right)} = e^{-\frac{(\log(d_n)-\mu)^2}{2\sigma^2}} e^{-\frac{c_n(\log(d_n)-\mu)}{\sigma^2 d_n}x} e^{O\left(\frac{c_n^2(\log(d_n)-\mu)}{d_n}x^2\right)}$$

By definition of d_n, $\frac{\sigma n}{\log(d_n)-\mu}e^{-\frac{(\log(d_n)-\mu)^2}{2\sigma^2}} = \sqrt{2\pi}$, so that

$$\begin{aligned}
\log\left(P\left(\frac{M_n - d_n}{c_n} < x\right)\right) &\to \frac{\log(d_n) - \mu}{\log(d_n) + \frac{c_n}{d_n}x + O\left(\frac{c_n^2}{d_n^2}x^2\right) - \mu} e^{-\frac{c_n(\log(d_n)-\mu)}{\sigma^2 d_n}x} e^{O\left(\frac{c_n^2(\log(d_n)-\mu)}{d_n^2}x^2\right)} \\
&= \left(1 - \frac{c_n}{d_n(\log(d_n) - \mu)}x + o\left(\frac{c_n}{d_n(\log(d_n) - \mu)}\right)\right) e^{-x} e^{o(1)x^2} \\
&\to e^{-x}
\end{aligned}$$

16

(a) We need to find c_n, d_n and ξ such that

$$nS(c_nx + d_n) = -\log(H_\xi(x))$$

For the Weibull distribution, the survival function is given by $S(x) = e^{-x^\tau}$, so we need to solve

$$ne^{-(c_nx+d_n)^\tau} \to -\log(H_\xi(x))$$

rearranging gives

$$(c_nx + d_n)^\tau \to -\log\left(n^{-1}\left(\log(H_\xi(x))\right)\right) = \log(n) - \log(\log(H_\xi(x)))$$

It follows that $c_nx + d_n = O\left(\log(n)^{\frac{1}{\tau}}\right)$. We consider three cases for the limit $\lim_{n\to\infty} \frac{d_n}{c_n}$. If $\frac{d_n}{c_n} \to \infty$, then

$$(c_nx + d_n)^\tau = d_n^\tau \left(1 + \frac{c_n}{d_n}x\right)^\tau = d_n^\tau + \tau c_n d_n^{\tau-1}x + o(c_n d_n^{\tau-1})$$

So we need

$$d_n^\tau + \tau c_n d_n^{\tau-1}x - \log(n) + \log(-\log(H_\xi(x))) = o(1)$$

for all x. It follows that $c_n d_n^{\tau-1} = O(1)$. Let $r = \lim_{n\to\infty} d_n^\tau - \log(n)$. Then we have $d_n^\tau - \log(n) = r + o(1)$. Similarly, if we let $s = \lim_{n\to\infty} c_n d_n^{\tau-1}$, then we have

$$r + \tau sx + o(1) = -\log(-\log(H_\xi(x)))$$

Thus,

$$-\log(-\log(H_\xi(x))) = r + \tau sx$$

This gives $\xi = 0$.

(b) We have $d_n = \theta\log(n)^{\frac{1}{\tau}}$ and $c_n = \frac{\theta^\tau}{\tau d_n^{\tau-1}} = \frac{\theta^\tau}{\tau\theta^{\tau-1}\log(n)^{\frac{\tau-1}{\tau}}} = \frac{\theta}{\tau}\log(n)^{\frac{1}{\tau}-1}$. Thus, we have

$$P(M_n > 100) = P\left(\frac{M_n - \theta\log(n)^{\frac{1}{\tau}}}{\frac{\theta}{\tau}\log(n)^{\frac{\tau}{\tau-1}}} > \frac{100 - \theta\log(n)^{\frac{1}{\tau}}}{\frac{\theta}{\tau}\log(n)^{\frac{1}{\tau}-1}}\right) = 1 - \exp\left(-\exp\left(-\frac{100 - \theta\log(n)^{\frac{1}{\tau}}}{\frac{\theta}{\tau}\log(n)^{\frac{1}{\tau}-1}}\right)\right)$$

The exact value is $1 - (1 - S(100))^n = 1 - \left(1 - e^{-\left(\frac{100}{\theta}\right)^\tau}\right)^n$

The probabilities are given in the following table:

	$n = 100$	$n = 1000$
Exact	0.8480498	1.0000000
GEV	0.8369774	0.9999943

17 The reinsurer needs to pay a claim in the next 100 years, if the maximum loss in those years exceeds \$20,000,000. That is, we want to calculate $P(M_{100} > 20)$. By the GEV, we have that $\frac{M_{100}-d_{100}}{c_{100}} \sim GEV(\xi)$. We are given that $c_{100} = 102^{\frac{1}{3}} = 4.67232872836$ and $d_{100} = 104^{\frac{2}{5}} = 6.40934037268$. Thus,

$$P(M_{100} > 20) = P\left(\frac{M_{100} - d_{100}}{c_{100}} > \frac{20 - 6.40934037268}{4.67232872836}\right) = P\left(\frac{M_{100} - d_{100}}{c_{100}} > 2.90875501649\right)$$

For the GEV with $\xi = 2$, we have $F(x) = e^{-(1+\xi x)^{\frac{1}{\xi}}}$, so $F(x) = e^{-(1+2x)^{\frac{1}{2}}}$. Therefore

$$P(M_{100} > 20) = 1 - H_2(2.90875501649) = 1 - e^{-\sqrt{1+2\times 2.90875501649}} = 0.926541613201$$

18

(a) We use the following code

```
library(QRM)
x<-rnorm(1000000)
estimate.GEV<-function(x,size){
     num_block<-floor(length(x)/size)
     M<-x[seq_len(num_block*size)]
     dim(M)<-c(size,num_block)
     maxima<-apply(M,2,max)
     return(fit.GEV(maxima))
}

GEV_ests<-rep(NA,100)
for(i in seq_len(100)){
     try(GEV_ests[i]<-estimate.GEV(x,i*1000)$par.ests[1])
     ## Use a try block as there are a few errors.
}

plot(GEV_ests,type='l')
```

We see that the estimates fluctuate more as the block size increases, as there are fewer block maxima in the samples.

(b) We use the following code

```
GEV_ests<-matrix(NA,100,100)

### This time we use 2j^2 as block sizes to include both small and
### large block sizes to demonstrate the problems with each.
for(i in seq_len(100)){
     x<-rnorm(1000000)
     for(j in seq_len(100)){
          try(GEV_ests[i,j]<-estimate.GEV(x,2*j^2)$par.ests[1])
          ## Use a try block as there are a few errors.
     }
}

library(ggplot2)
library(reshape)
library(dplyr)
ggplot(GEV_ests%>%melt(),
        mapping=aes(x=2*X2^2,y=value,group=as.factor(X2)))+
     geom_violin()+
     scale_x_log10()
```

We see that the estimates fluctuate more as the block size increases, as there are fewer block maxima in the samples.

19 Let $X \sim G_{\xi,\beta}$. We will first show that $X - d|X > d \sim G_{\xi,\beta^*}$ where $\beta^* = \beta + \xi d$.
We have $S_{X-d|X>d}(x) = \frac{S_X(x+d)}{S_X(d)}$. For $\xi \neq 0$, this gives

$$
\begin{aligned}
S_{X-d|X>d}(x) &= \frac{\left(1+\xi\frac{x+d}{\beta}\right)^{\frac{1}{\xi}}}{\left(1+\xi\frac{d}{\beta}\right)^{\frac{1}{\xi}}} \\
&= \left(\frac{1+\xi\frac{x+d}{\beta}}{1+\xi\frac{d}{\beta}}\right)^{\frac{1}{\xi}} \\
&= \left(\frac{\beta+\xi(x+d)}{\beta+\xi d}\right)^{\frac{1}{\xi}} \\
&= \left(1+\frac{\xi x}{\beta+\xi d}\right)^{\frac{1}{\xi}} \\
&= 1 - G_{\xi,(\beta+\xi d)}(x)
\end{aligned}
$$

For $\xi = 0$ it gives

$$
\begin{aligned}
S_{X-d|X>d}(x) &= \frac{e^{-\frac{x+d}{\beta}}}{e^{-\frac{d}{\beta}}} \\
&= e^{-\frac{x}{\beta}} \\
&= 1 - G_{0,\beta}(x)
\end{aligned}
$$

Thus, $e(d) = \mathbb{E}(X - d|X > d)$ is the expectation of a GPD with parameters ξ and β^*. Since β is a scale parameter, this is β^* times the expectation of a GPD with parameters ξ and 1. That is, we have $e(d) = (\beta + \xi d)e_{\xi,1}$, which is clearly a linear function, whenever $e_{\xi,1}$ is finite.

20

(a) We use the following code to calculate the empirical MEL:

```
### Simulate data
x<-1/rgamma(100000,shape=4)
### sort data
x<-sort(x)
### Calculate Mean Excess Loss
MEL<-rev(cumsum(rev(x))/seq_along(x))

### Fit a linear model on the linear part.
lm(y~x,data.frame(x=x[x>0.4&x<3],y=MEL[x>0.4&x<3]))
```

(b) We plot the empirical MEL on a graph and see that the linear approximation becomes reasonable at a threshold of about 0.25.

(c) Fitting a linear model on the linear part of the distribution gives $e(d) = 1.2966d + 0.1016$. By Question 19, we see that this corresponds to the GPD parameters $\frac{\xi^2}{1-\xi} = 1.2966$ or $\xi = \frac{\sqrt{1.2966^2+4\times 1.2966}-1.2966}{2} = 0.662002594825$ and $\beta = 0.1016\frac{1-\xi}{\xi} + \xi d = 0.0518737186746 + 0.662002594825d$. In particular $X - 0.3|X > 0.3$ approximately follows a GPD with $\xi = 0.662002594825$ and $\beta = 0.0518737186746 + 0.662002594825 \times 0.3 = 0.250474497123$. Empirically $S_X(0.3) \approx 0.42453$. Using the GPD approximation,

$$S_X(1.3) = S_X(0.3)S_{X-0.3|X>0.3}(1) \approx 0.42453\times\left(1 + 0.662002594825\frac{1}{0.0518737186746}\right)^{-\frac{1}{0.662002594825}} = 0.008088376647$$

The empirical estimate for this probability is 0.00766. The actual value is 0.007949451.

21 Suppose the distribution of X has $e(d) = a + \xi d$ for some a and ξ. We have $e(d) = \frac{\int_d^\infty S(x)\,dx}{S(d)}$, so

$$\begin{aligned}\frac{de(d)}{dd} &= \frac{-S(d)}{S(d)} - \frac{S'(d)\int_d^\infty S(x)\,dx}{S(d)^2} \\ &= \lambda(d)e(d) - 1\end{aligned}$$

where λ is the hazard rate function of X. Substituting $e(d) = a + \xi d$, and thus $\frac{de(d)}{dd} = \xi$ gives

$$\lambda(d) = \frac{\xi + 1}{a + \xi d}$$

Since the hazard rate is fixed, this determines the distribution up to a scale. Thus, the only distributions with linear MEL functions are the Generalised Poisson distribution.

22 Since the GPD with $\xi = 0.4$ and $\beta = 300$ is a good approximation for $X - 4200|X > 4200$, we have $S_{4200}(x) = \frac{S_X(4200+x)}{S_X(4200)} = \frac{S_X(4200+x)}{0.05}$, so the 99th percentile of X is the solution to $S_X(y) = 0.01$, so $S_{4200}(y - 4200) = \frac{0.01}{0.05} = 0.2$. That is, we need to solve

$$\begin{aligned}
\left(1 + 0.4\frac{y - 4200}{300}\right)^{-\frac{1}{0.4}} &= 0.2 \\
0.4\frac{y - 4200}{300} &= 5^{0.4} - 1 \\
y &= 750\left(5^{0.4} - 1\right) + 4200 \\
&= 4877.74045404
\end{aligned}$$

For the TVaR, we have that

$$\mathrm{TVaR}_{0.99}(X) = \mathrm{VaR}_{0.99}(X) + e(\mathrm{VaR}_{0.99}(X)) = 4877.74045404 + \frac{300 + 0.4\,(4877.74045404 - 4200)}{1 - 0.4} = 5829.5674234$$

SN2 5.4.4 The Hill Estimator

23 Since F is in the MDA of a Fréchet distribution, the survival function of X is of the form $S(x) = L(x)x^{-\frac{1}{\xi}}$. Thus $S_{\log(X)}(x) = S_X(e^x) = L(e^x)e^{-\frac{x}{\xi}}$. Since L is slowly varying, we have that $\frac{L(e^{x+t})}{L(x)} \to 1$ for any fixed t as $x \to \infty$, the mean excess loss of $\log(x)$ is then given by

$$e(d) = \frac{\int_d^\infty S(e^x)\,dx}{S(e^d)} = \frac{\int_d^\infty L(e^x)e^{-\frac{x}{\xi}}\,dx}{L(e^d)e^{-\frac{d}{\xi}}} = \int_d^\infty \frac{L(e^x)}{L(e^d)}e^{-\frac{x-d}{\xi}}\,dx = \int_0^\infty \frac{L(e^{x+d})}{L(e^d)}e^{-\frac{x}{\xi}}\,dx$$

For each x, as $d \to \infty$, $\frac{L(e^{x+d})}{L(e^d)} \to 1$, so

$$\int_0^\infty \frac{L(e^{x+d})}{L(e^d)}e^{-\frac{x}{\xi}}\,dx \to \int_0^\infty e^{-\frac{x}{\xi}}\,dx = \xi$$

24

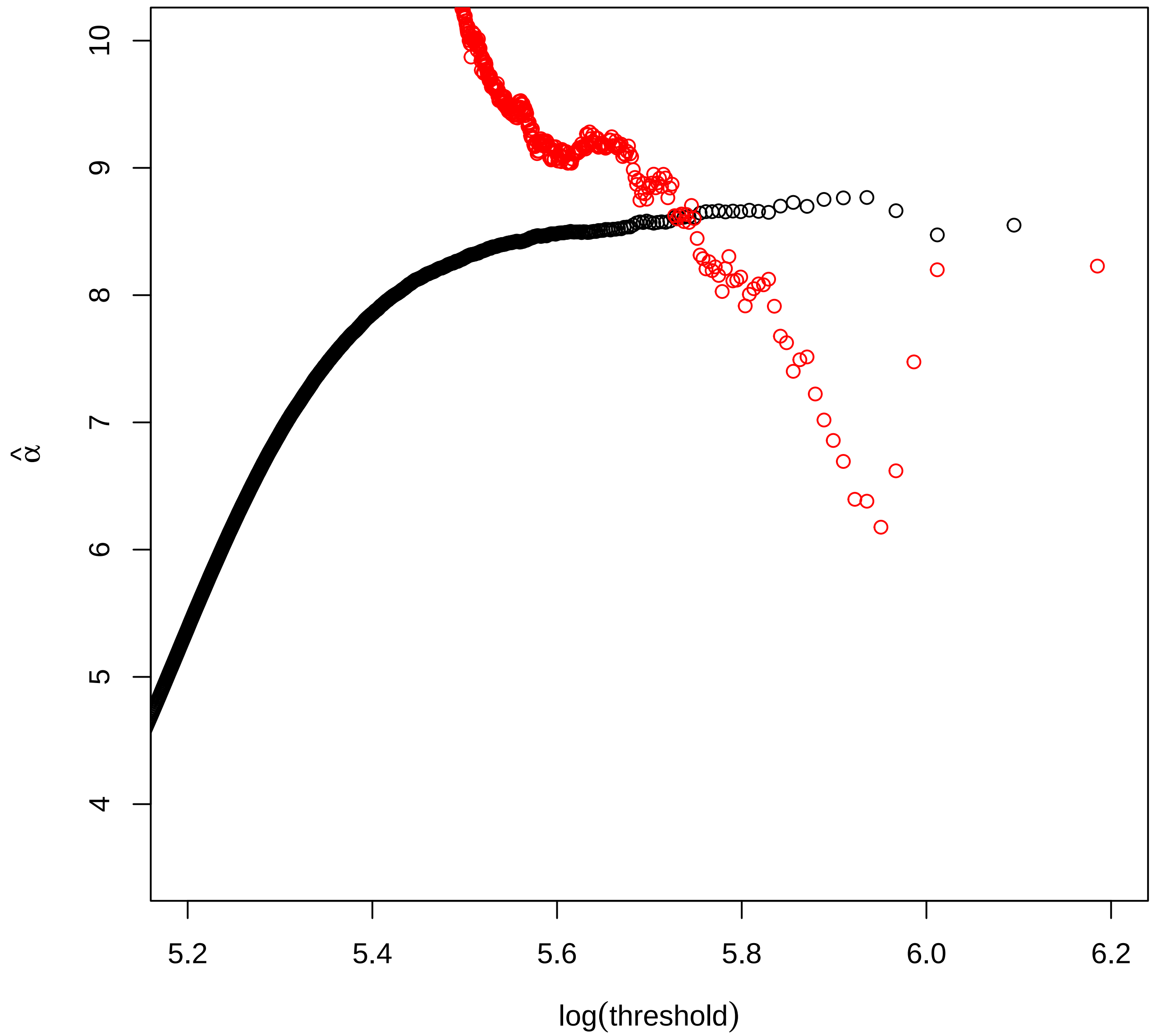
We use the following code:

```
set.seed(1234)
n<-1000000
x<-200/rweibull(n,shape=8.7)
x<-sort(x)
lx<-log(x)
Hill.alpha<-1/(rev((cumsum(rev(lx))/(1+seq_along(lx))))[-1]-lx[-n])
GPD.MLE<-rep(NA,n)
for(i in seq_len(1000)*n/2000+n/2-1){
    print(i)
    GPD.MLE[i]<-fit.GPD(x,nextremes=n-i)$par.ests["xi"]
}


### Only plot every 1000th point to reduce computation.
plot(lx[seq_len(9999)*1000],Hill.alpha[seq_len(9999)*1000],
     xlab=expression(log(threshold)),ylab=expression(hat(alpha)),
     ylim=c(3.5,10))

### Compare MLE. Remember that alpha=1/xi.
points(lx[which(!is.na(GPD.MLE))],1/GPD.MLE[which(!is.na(GPD.MLE))],col="red")
```

Computing the MLE takes a while, so we only compute for a sample of thresholds.



We see that for the thresholds between about 5.5 and 5.8, the Hill estimate is relatively stable, and not too far from the true value 8.7. For larger thresholds, the estimates start to become unstable due to the small sample size.

The MLE is close to the truth in a narrow range between about 5.55 and 5.7.

7.3 Compound Distributions

25 Let Λ be the random Poisson parameter. We have

$$\begin{aligned} P(X=n) &= \mathbb{E}(P(X=n|\Lambda)) \\ &= \mathbb{E}\left(e^{-\Lambda}\frac{\Lambda^n}{n!}\right) \\ &= \frac{\mathbb{E}\left(e^{-\Lambda}\Lambda^n\right)}{n!} \end{aligned}$$

where $\Lambda \sim \text{Gamma}(\alpha = 0.4, \theta = 3)$. The MGF of the Gamma distribution is $M_\Lambda(t) = \mathbb{E}\left(e^{\Lambda t}\right)$. We thus get that $M'_\Lambda(t) = \mathbb{E}\left(\Lambda e^{\Lambda t}\right)$ and more generally $M_\Lambda^{(n)}(t) = \mathbb{E}\left(\Lambda^n e^{\Lambda t}\right)$ Thus, we have $\mathbb{E}\left(e^{-\Lambda}\Lambda^n\right) = M_\Lambda^{(n)}(-1)$.

For the Gamma distribution with shape α and scale θ, we have $M_\Lambda(t) = (1-\theta t)^{-\alpha}$. Differentiating gives $M_\Lambda^{(n)}(t) = (-\theta)^n n!\binom{-\alpha}{n}(1-\theta t)^{-\alpha-n}$. Thus we have

$$\mathbb{E}\left(e^{-\Lambda}\Lambda^n\right) = M_\Lambda^{(n)}(-1) = (-\theta)^n n!\binom{-\alpha}{n}(1+\theta)^{-\alpha-n}$$

Thus

$$P(X=n) = \binom{-\alpha}{n}(1+\theta)^{-\alpha}\left(-\frac{\theta}{1+\theta}\right)^n = \binom{n+\alpha}{n}(1+\theta)^{-\alpha}\left(\frac{\theta}{1+\theta}\right)^n$$

7.1 Compound Distributions

26

Let P be the pgf of the secondary distribution. Let Q be the pgf of the primary distribution. Let R be the pgf of the compound distribution.

Note that $Q'(z) = \sum_{n=1}^{\infty} nq_n z^{n-1} = \sum_{n=1}^{\infty}(na+b)q_{n-1}z^{n-1} = azQ'(z)+(a+b)Q(z)$ so $Q'(z) = \frac{(a+b)}{(1-az)}Q(z)$

We therefore have $R'(z) = P'(z)Q'(P(z)) = P'(z)\frac{a+b}{1-aP(z)}Q(P(z))$

We therefore get

$$(1 - aP(z))R'(z) = (a + b)P'(z)R(z)$$

Letting the probabilities of the primary distribution be p_n and the probabilities of the compound distribution be r_n, and writing this equation in power series then equating coefficients of z^{k-1} gives

$$\left(1 - a\sum_{n=0}^{\infty} p_n z^n\right)\left(\sum_{m=0}^{\infty} mr_m z^{m-1}\right) = (a + b)\left(\sum_{n=1}^{\infty} np_n z^{n-1}\right)\left(\sum_{m=0}^{\infty} r_n z^n\right)$$

$$(1 - ap_0)kr_k - a\sum_{n=1}^{k-1} p_n(k - n)r_{k-n} = (a + b)\sum_{n=1}^{k} np_n r_{k-n}$$

$$(1 - ap_0)kr_k = a\sum_{n=1}^{k} p_n(k - n)r_{k-n} + (a + b)\sum_{n=1}^{k} np_n r_{k-n}$$

$$= \sum_{n-1}^{k} \left(a(k - n) + (a + b)n\right) p_n r_{k-n}$$

$$r_k = \frac{\sum_{n-1}^{k} \left(a + b\frac{n}{k}\right) p_n r_{k-n}}{1 - ap_0}$$

27

From first principles:

Let N be the primary distribution. Since a sum of i.i.d. Poisson random variables is Poisson, the conditional distribution of the compound distribution is $S|N=n \sim Po(n\lambda_1)$. Thus we calculate the probabilities for the compound distribution by $P(S=k)=\mathbb{E}(P(S=k|N))$.

$$\begin{aligned}
P(S=0) &= \mathbb{E}(P(S=k|N)) \\
&= e^{-\lambda_1}\sum_{n=0}^{\infty}\frac{\lambda_1^n}{n!}e^{-n\lambda_2} \\
&= e^{-\lambda_1}\sum_{n=0}^{\infty}\frac{\left(e^{-\lambda_2}\lambda_1\right)^n}{n!} \\
&= e^{-\lambda_1\left(1-e^{-\lambda_2}\right)} \\
P(S=1) &= \mathbb{E}(P(S=k|N)) \\
&= e^{-\lambda_1}\sum_{n=0}^{\infty}\frac{\lambda_1^n}{n!}e^{-n\lambda_2}\lambda_2 n \\
&= \lambda_2 e^{-\lambda_1}\sum_{n=1}^{\infty}\frac{\left(e^{-\lambda_2}\lambda_1\right)^n}{(n-1)!} \\
&= \lambda_2 e^{-\lambda_1}e^{-\lambda_2}\lambda_1\sum_{m=0}^{\infty}\frac{\left(e^{-\lambda_2}\lambda_1\right)^m}{m!} \\
&= \lambda_1\lambda_2 e^{-\lambda_1-\lambda_2+\lambda_1 e^{-\lambda_2}} \\
P(S=2) &= \mathbb{E}(P(S=k|N)) \\
&= e^{-\lambda_1}\sum_{n=0}^{\infty}\frac{\lambda_1^n}{n!}e^{-n\lambda_2}\frac{(n\lambda_2)^2}{2} \\
&= e^{-\lambda_1}\sum_{n=0}^{\infty}\frac{\lambda_1^n}{n!}e^{-n\lambda_2}\frac{(n\lambda_2)^2}{2} \\
&= \frac{\lambda_2^2}{2}e^{-\lambda_1}\sum_{n=1}^{\infty}n\frac{\left(e^{-\lambda_2}\lambda_1\right)^n}{(n-1)!} \\
&= \frac{1}{2}\lambda_1\lambda_2^2 e^{-\lambda_1}e^{-\lambda_2}\lambda_1\sum_{m=0}^{\infty}(m+1)\frac{\left(e^{-\lambda_2}\lambda_1\right)^m}{m!} \\
&= \frac{1}{2}\lambda_2^2 e^{-\lambda_1}e^{-\lambda_2}\lambda_1\left(\sum_{m=0}^{\infty}m\frac{\left(e^{-\lambda_2}\lambda_1\right)^m}{m!}+\sum_{m=0}^{\infty}\frac{\left(e^{-\lambda_2}\lambda_1\right)^m}{m!}\right) \\
&= \frac{\lambda_1\lambda_2^2}{2}e^{-\lambda_1-\lambda_2}\left(e^{-\lambda_2}\lambda_1 e^{e^{-\lambda_2}\lambda_1}+e^{e^{-\lambda_2}\lambda_1}\right) \\
&= \frac{\lambda_1\lambda_2^2}{2}e^{-\lambda_1-\lambda_2}\left(e^{-\lambda_2}\lambda_1+1\right)e^{e^{-\lambda_2}\lambda_1}
\end{aligned}$$

Using the recurrence:

The p.g.f. of the compound Poisson distribution is $P(z) = e^{\lambda_1\left(e^{\lambda_2(z-1)}-1\right)}$. This gives $p_0 = P(0) = e^{\lambda_1\left(e^{-\lambda_2}-1\right)}$. Now the recurrence is

$$f_S(n) = \sum_{i=1}^{n} \lambda_1 \frac{i}{n} e^{-\lambda_2} \frac{\lambda_2^i}{i!} f_S(n-i)$$

Thus we get

$$\begin{aligned}
P(S=1) &= \lambda_1 \frac{1}{1} e^{-\lambda_2} \frac{\lambda_2}{1!} f_S(0) \\
&= \lambda_1 \lambda_2 e^{-\lambda_2} e^{\lambda_1\left(e^{-\lambda_2}-1\right)} \\
P(S=2) &= \lambda_1 \left(\frac{1}{2} e^{-\lambda_2} \frac{\lambda_2}{1!} f_S(1) + \frac{2}{2} e^{-\lambda_2} \frac{\lambda_2^2}{2!} f_S(0) \right) \\
&= \frac{\lambda_1 \lambda_2^2}{2} e^{-\lambda_1-\lambda_2} \left(\lambda_1 e^{-\lambda_2} + 1\right) e^{\lambda_1\left(e^{-\lambda_2}-1\right)}
\end{aligned}$$

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Binomial has pgf $P(s) = (1 - p + ps)^n$, negative binomial has pgf $P(s) = (1 + \beta - \beta s)^{-r}$. The geometric distribution has pgf $P(s) = \frac{1}{1+\beta-\beta s}$. The compound binomial-geometric therefore has pgf

$$P(s) = \left(1 - p + \frac{p}{1 + \beta_1 - \beta_1 s}\right)^n = \left(\frac{1 + \beta_1 - \beta_1 p - (1-p)\beta_1 s}{1 + \beta_1 - \beta_1 s}\right)^n$$

while the compound negative binomial-geometric has pgf

$$\left(1 + \beta_2 - \frac{\beta_2}{1 + \beta_3 - \beta_3 s}\right)^{-r} = \left(\frac{(1+\beta_2)(1+\beta_3) - \beta_2 - \beta_3(1+\beta_2)s}{1 + \beta_3 - \beta_3 s}\right)^{-r} = \left(\frac{1 + \beta_3 - \beta_3 s}{(1+\beta_2)(1+\beta_3) - \beta_2 - \beta_3(1+\beta_2)s}\right)^{-r}$$

We see that these are equal by setting $\beta_1 = \beta_3(1 + \beta_2)$ and $p = \frac{\beta_2 \beta_3}{\beta_1}$.

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pgf of Poisson is $P(s) = e^{\lambda(s-1)}$, pgf of logarithmic is $\frac{\log(1-as)}{\log(1-a)}$, so the pgf of the compound Poisson-logarithmic is $e^{\lambda\left(\frac{\log(1-as)}{\log(1-a)}-1\right)} = e^{\frac{\lambda}{\log(1-a)}(\log(1-as)-\log(1-a))} = \left(\frac{1-as}{1-a}\right)^{\frac{\lambda}{\log(1-a)}}$. Setting $\lambda = -r\log(1-a)$, and $a = \frac{\beta}{1+\beta}$, this becomes $P(s) = (1+\beta-\beta s)^{-r}$, which is the pgf of the negative binomial distribution.

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Let secondary distributions have pgf $Q(s)$ and $R(S)$. Then compound distributions have pgf $e^{\lambda_1(Q(s)-1)}$ and $e^{\lambda_2(R(s)-1)}$, so since they are independent, we have that their sum has pgf $P(s) = e^{\lambda_1(Q(s)-1)}e^{\lambda_2(R(s)-1)} = e^{\lambda_1(Q(s)-1)+\lambda_2(R(s)-1)}$. Setting $\lambda = \lambda_1 + \lambda_2$, we get

$$P(s) = e^{\lambda\left(\frac{\lambda_1 Q(s)+\lambda_2 R(s)}{\lambda}-1\right)}$$

so this is a compound Poisson distribution where the secondary distribution has pgf $\frac{\lambda_1 Q(s)+\lambda_2 R(s)}{\lambda_1+\lambda_2}$. This is the pgf of a mixture of the distributions with pgf Q and R.

31 (a) If the secondary distribution has raw moments μ_1, μ_2' and μ_3', then we have

$$\begin{aligned}
\mathbb{E}(S) &= \mathbb{E}(\mathbb{E}(S|N)) = \mathbb{E}(N\mathbb{E}(X)) = \mathbb{E}(N)\mathbb{E}(X)\\
&= \lambda\mu_1\\
\mathbb{E}(S^2) &= \mathbb{E}(\mathbb{E}(S^2|N)) = \mathbb{E}(\mathbb{E}\left((X_1+\cdots+X_N)^2\right))\\
&= \mathbb{E}(N\mathbb{E}(X_i^2) + N(N-1)\mathbb{E}(X_iX_j))\\
&= \mathbb{E}\left(N^2\mathbb{E}(X_i)^2 + N\operatorname{Var}(X_i))\right)\\
&= \mathbb{E}\left(N^2\right)\mathbb{E}(X_i)^2 + \mathbb{E}(N)\operatorname{Var}(X_i))\\
&= \lambda(\lambda+1)\mu_1^2 + \lambda(\mu_2' - \mu_1^2)\\
&= \lambda^2\mu_1^2 + \lambda\mu_2'\\
\operatorname{Var}(S) &= \mathbb{E}\left(N^2\right)\mathbb{E}(X_i)^2 + \mathbb{E}(N)\operatorname{Var}(X_i)) - \mathbb{E}\left(N\right)^2\mathbb{E}(X_i)^2\\
&= \operatorname{Var}(N)\mathbb{E}(X_i)^2 + \mathbb{E}(N)\operatorname{Var}(X_i)\\
&= \lambda\mu_2'\\
\mathbb{E}(S^3) &= \mathbb{E}(\mathbb{E}(S^3|N)) = \mathbb{E}\left(\mathbb{E}\left((X_1+\cdots+X_N)^3\right)\right)\\
&= \mathbb{E}\left(\sum_{i,j,k=1}^{N}\mathbb{E}\left(X_iX_jX_k\right)\right)\\
&= \mathbb{E}\left(N\mathbb{E}(X_i^3) + N(N-1)\mathbb{E}(X_i^2)\mathbb{E}(X_j) + N(N-1)(N-2)\mathbb{E}\left(X_i\right)^3\right)\\
&= \mathbb{E}(N)\mu_3' + \mathbb{E}(N(N-1))\mu_1\mu_2' + \mathbb{E}(N(N-1)(N-2))\mu_1^3\\
&= \lambda\mu_3' + \lambda^2\mu_1\mu_2' + \lambda^3\mu_1^3
\end{aligned}$$

The centralised third moment is therefore

$$\lambda\mu_3' + \lambda^2\mu_1\mu_2' + \lambda^3\mu_1^3 - 3\lambda\mu_1(\lambda^2\mu_1^2 + \lambda\mu_2') + 2\lambda^3\mu_1^3 = \lambda\mu_3' - 2\lambda^2\mu_1\mu_2'$$

The skewness is

$$\frac{\lambda\mu_3' - 2\lambda^2\mu_1\mu_2'}{(\lambda\mu_2')^{\frac{3}{2}}} = \frac{\mu_3'}{\mu_2'^{\frac{3}{2}}\sqrt{\lambda}} - 2\sqrt{\lambda}\frac{\mu_1}{\sqrt{\mu_2'}}$$

(b) The ETNB distribution has $\mu_1 = \frac{r\beta}{1-(1+\beta)^{-r}}$, $\mu_2' = \frac{r(r+1)\beta^2+r\beta}{1-(1+\beta)^{-r}}$ and $\mu_3' = \frac{r(r+1)(r+2)\beta^3+3r(r+1)\beta^2+r\beta}{1-(1+\beta)^{-r}}$. Thus, the skewness of the compound Poisson-ETNB distribution is

$$\frac{\frac{r(r+1)(r+2)\beta^3+3r(r+1)\beta^2+r\beta}{1-(1+\beta)^{-r}}}{\left(\frac{r(r+1)\beta^2+r\beta}{1-(1+\beta)^{-r}}\right)^{\frac{3}{2}}\sqrt{\lambda}} - 2\sqrt{\lambda}\frac{\frac{r\beta}{1-(1+\beta)^{-r}}}{\sqrt{\frac{r(r+1)\beta^2+r\beta}{1-(1+\beta)^{-r}}}} = \frac{r(r+1)(r+2)\beta^3+3r(r+1)\beta^2+r\beta}{(r(r+1)\beta^2+r\beta)^{\frac{3}{2}}\sqrt{\lambda}\sqrt{1-(1+\beta)^{-r}}} - \frac{2\sqrt{\lambda}r\beta}{\sqrt{1-(1+\beta)^{-r}}\sqrt{r(r+1)\beta^2+r\beta}}$$

$$= \sqrt{\frac{\lambda r^2\beta^2}{(r(r+1)\beta^2+r\beta)\left(1-(1+\beta)^{-r}\right)}}\left(\frac{(r+1)(r+2)\beta^2+3(r+1)\beta+1}{\lambda\left(r(r+1)\beta^2+r\beta\right)} - 2\right)$$

$$= \sqrt{\frac{\lambda r\beta}{((r+1)\beta+1)\left(1-(1+\beta)^{-r}\right)}}\left(\frac{(r+1)(r+2)\beta^2+3(r+1)\beta+1}{\lambda\left(r(r+1)\beta^2+r\beta\right)} - 2\right)$$

a) The M.G.F. of the gamma distribution is $M(t) = (1-\theta t)^{-\alpha} = \left((1-\theta t)^{-\frac{\alpha}{n}}\right)^n$ for any positive integer n. This is the M.G.F. of a gamma distribution.

b) The M.G.F. of the inverse gamma distribution for is given by

$$M(-t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{-tx} \frac{\theta^\alpha e^{-\frac{\theta}{x}}}{x^{\alpha+1}\Gamma(\alpha)}\,dx = \mathbb{E}(e^{-tX}) = \int_0^\infty \frac{\theta^\alpha e^{-tx-\frac{\theta}{x}}}{x^{\alpha+1}\Gamma(\alpha)}\,dx$$

$-\frac{1}{\theta} < t < 0$ is $M(t) = (1+\theta t)^\alpha = \left((1+\theta t)^{\frac{\alpha}{n}}\right)^n$ Thus, $(1+\theta t)^{\frac{\alpha}{n}}$ is the M.G.F. of another inverse gamma distribution.

c) The M.G.F. of the inverse Gaussian distribution is $M(t) = \exp\left(\frac{\theta}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 t}{\theta}}\right)\right)$. This gives

$$\begin{aligned} M(t)^{\frac{1}{n}} &= \exp\left(\frac{\theta}{n\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\theta}t}\right)\right) \\ &= \exp\left(\frac{\left(\frac{\theta}{n^2}\right)}{\left(\frac{\mu}{n}\right)}\left(1-\sqrt{1-\frac{2\left(\frac{\mu}{\theta}\right)^2}{\left(\frac{\theta}{n}\right)t}}\right)\right) \end{aligned}$$

This is the M.G.F. of another inverse gaussian distribution.

d) Since the binomial distribution is discrete, it is easier to use its P.G.F. $P(z) = (1-p+pz)^n$. To divide as a sum of m i.i.d. random variables, where $m > n$, we have

$$\begin{aligned} P(z)^{\frac{1}{m}} &= (1-p+pz)^{\frac{n}{m}} \\ &= (1-p)^{\frac{n}{m}}\left(1+\frac{p}{1-p}z\right)^{\frac{n}{m}} \\ &= (1-p)^{\frac{n}{m}}\sum_{k=0}^{\infty}\binom{\frac{n}{m}}{k}\left(\frac{p}{1-p}\right)^k z^k \end{aligned}$$

In particular, we can see that the coefficient of z is negative, so this is not the P.G.F. of a probability distribution.

Easier way: The binomial distribution has finite support. Since it is discrete, any division must also be discrete. Consider the $(n+1)th$ division. If this has only 1 possible value, the sum would also have one possible value. If it has two, then there are at least $n+2$ possible values for the sum.

9.3 The Compound Model for Aggregate Claims

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We have

n	$P(N=n)$	Z	$P(A>130\|N=n)$
0	0.4	∞	0
1	0.3	0.8571	0.1957
2	0.2	-1.414	0.9214
3	0.1	-2.804	0.9975

So the probability is $0+0.3\times 0.1957+0.2\times 0.9214+0.1\times 0.9975 = 0.0587+0.1843+0.0997 = 0.3427$.

9.4 Analytic Results

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The severity is exponential with mean θ. The frequency is negative binomial with parameters $r = 2$ and β. The aggregate severity of n losses therefore follows a gamma distribution with $\alpha = n$. We therefore have that the probability that the aggregate loss is zero is $p_0 = \frac{1}{(1+\beta)^2}$, while if it is non-zero, the pdf of the aggregate loss is

$$\begin{aligned} f(x) &= (1+\beta)^{-2} \sum_{n=1}^{\infty} (n+1) \left(\frac{\beta}{1+\beta} \right)^n \frac{x^{n-1}}{\theta^n (n-1)!} e^{-\frac{x}{\theta}} \\ &= (1+\beta)^{-2} e^{-\frac{x}{\theta}} \frac{\beta}{\theta(1+\beta)} \sum_{n=1}^{\infty} \frac{(n+1)}{(n-1)!} \left(\frac{x\beta}{\theta(1+\beta)} \right)^{n-1} \end{aligned}$$

We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(n+1)a^{n-1}}{(n-1)!} &= \frac{1}{a} \frac{d}{da} \left(\sum_{n=1}^{\infty} \frac{a^{n+1}}{(n-1)!} \right) \\ &= \frac{1}{a} \frac{d}{da} \left(a^2 \sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!} \right) \\ &= \frac{1}{a} \frac{d}{da} \left(a^2 e^a \right) \\ &= (a+2)\, e^a \end{aligned}$$

Substituting this into the equation above gives

$$\begin{aligned} f(x) &= (1+\beta)^{-2} e^{-\frac{x}{\theta}} \frac{\beta}{\theta(1+\beta)} \left(\left(\frac{x\beta}{\theta(1+\beta)} \right) + 2 \right) e^{\left(\frac{x\beta}{\theta(1+\beta)} \right)} \\ &= (1+\beta)^{-4} \theta^{-2} \beta \left(x\beta + 2 \right) e^{-\frac{x}{\theta(1+\beta)}} \end{aligned}$$

This is a mixture of gamma distributions.

Alternatively, we can obtain this result using moment generating functions. For an aggregate loss model whose frequency distribution has probability generating function $P_N(z)$, and whose severity distribution has moment generating function $M_X(t)$, the aggregate loss model has moment generating function $M_S(t) = P_N(M_X(t))$.

In the case of the negative binomial gamma distibution, $P_N(z) = (1+\beta-\beta z)^{-r}$ and $M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}$. This gives

$$M_S(t) = \left(1+\beta-\beta\left(\frac{1}{1-\theta t}\right)^{\alpha}\right)^{-r}$$

In the particular case $r = 2, \alpha = 1$ gives

$$\begin{aligned}
M_S(t) &= \left(1+\beta-\beta\frac{1}{1-\theta t}\right)^{-2} = \left(\frac{(1+\beta)(1-\theta t)-\beta}{1-\theta t}\right)^{-2} = \left(\frac{1-\theta t}{1-(1+\beta)\theta t}\right)^{2} \\
&= \left(\frac{1}{1+\beta} + \left(\frac{\beta}{1+\beta}\right)\frac{1}{1-(1+\beta)\theta t}\right)^{2} \\
&= \frac{1}{(1+\beta)^2} + 2\frac{\beta}{(1+\beta)^2}\frac{1}{1-(1+\beta)\theta t} + \frac{\beta^2}{(1+\beta)^2}\frac{1}{(1-(1+\beta)\theta t)^2}
\end{aligned}$$

This is a mixture of a point mass at 0 with probability $(1+\beta)^{-2}$; an exponential distribution with mean $(1+\beta)\theta$ with probability $\frac{2\beta}{(1+\beta)^2}$; and a gamma distribution with $\theta = (1+\beta)\theta, \alpha = 2$ with probability $\left(\frac{\beta}{(1+\beta)}\right)^2$.

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By the general result, the compound negative binomial-exponential with $r = 15$, $\beta = 2.4$ and $\theta = 3000$ is the same a compound binomial-exponential with $n = 15$, $p = \frac{2.4}{3.4} = \frac{12}{17}$ and $\theta = 3000 \times 2.4 = 7200$.

If there are n claims, the aggregate loss follows a gamma distribution with $\alpha = n$ and $\theta = 10200$. The expected payment on the stop-loss insurance is then $10200 \int_{\frac{204000}{10200}}^{\infty} \left(x - \frac{204000}{10200}\right) \frac{x^{n-1}e^{-x}}{(n-1)!}dx$.

We have

$$\int_a^\infty \frac{x^n e^{-x}}{n!}dx = e^{-a}\left(1 + a + \frac{a^2}{2} + \ldots + \frac{a^n}{n!}\right)$$

so the expected payment on the stop-loss insurance if there are n claims is

$$\begin{aligned}
10200 \int_{20}^{\infty} (x-20)\frac{x^{n-1}e^{-x}}{(n-1)!}dx &= 7200\left(\int_{20}^{\infty}\frac{x^n e^{-x}}{(n-1)!}dx - 20\int_{20}^{\infty}\frac{x^{n-1}e^{-x}}{(n-1)!}dx\right)\\
&= 10200\left(n\int_{20}^{\infty}\frac{x^n e^{-x}}{n!}dx - 20\int_{20}^{\infty}\frac{x^{n-1}e^{-x}}{(n-1)!}dx\right)\\
&= 10200e^{-20}\left(n\left(1+20+\frac{20^2}{2!}+\cdots+\frac{20^n}{n!}\right) - 20\left(1+20+\frac{20^2}{2!}+\cdots+\frac{20^{n-1}}{(n-1)!}\right)\right)\\
&= 10200e^{-20}\left(n + 20(n-1) + \frac{20^2(n-2)}{2!} + \cdots + \frac{20^{n-1}}{(n-1)!}\right)
\end{aligned}$$

The overall expected payment on the stop-loss insurance is therefore

$$\begin{aligned}
&10200e^{-20}\sum_{n=1}^{15}\left(\binom{15}{n}\frac{12^n 5^{15-n}}{17^{15}}\right)\left(n + 20(n-1) + \frac{20^2(n-2)}{2!} + \cdots + \frac{20^{n-1}}{(n-1)!}\right)\\
&= 10200e^{-20}\sum_{n=1}^{15}\left(\binom{15}{n}\frac{12^n 5^{15-n}}{17^{15}}\right)\sum_{k=0}^{n}(n-k)\frac{20^k}{k!}
\end{aligned}$$

We can evaluate this sum in R using matrix operations:

```
expseries <-20^(0:14)/factorial(0:14)  #The terms in k
nvect<-dbinom(1:15,size=15,prob=12/17)  #The terms in n
nminusk<-pmax(rep(1,15)%*%t(1:15)-(1:15)*rep(1,15),0)  #the n-k term
t(expseries)%*%nminusk%*%nvect/exp(20)*10200  #The expected payment
```

This gives the expected payment on the stop-loss insurance as \$137.17.

9.5 Computing the Aggregate Claims Distribution

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ETNB $r = -0.6$ $\beta = 7$,
$a = \frac{\beta}{1+\beta} = \frac{7}{8}$, $b = \frac{(r-1)\beta}{1+\beta} = -1.4$
$q_1 = \frac{-0.6\times 7}{8(8^{-0.6}-1)} = 0.7365057$ $q_2 = \left(\frac{7}{8} - \frac{1.4}{2}\right) q_1 = 0.1288885$ $q_3 = \left(\frac{7}{8} - \frac{1.4}{3}\right) q_2 = 0.05262947$
$(n+1)\left(\frac{3}{4}\right)^n$ 0.0625 0.0625 0.09375 0.1054687890625 0.1054687890625

$p_0 = 0.0625$

$p_1 = 0.09375 * 0.7365057 = 0.06904741$

$p_2 = 0.09375 * 0.1288885 + 0.1054687890625 * 0.7365057^2 = 0.06929383$

$p_3 = 0.09375 * 0.05262947 + 0.1054687890625 * (0.7365057 * 0.1288885 * 2) + 0.1054687890625 * 0.7365057^3 = 0.06709359$

So the total probability that the aggregate loss is at most 3 is $0.0625 + 0.06904741 + 0.06929383 + 0.06709359 = 0.2679348$.

9.6 The Recursive Method

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The recurrence formula gives $f_S(n) = \frac{\sum_{i=1}^{n}\left(a+\frac{bi}{n}\right)f_X(i)f_S(n-i)}{1-af_X(0)}$ $f_S(n) = \sum_{i=1}^{n} \frac{2.4i}{n}\binom{i+9}{i}\left(\frac{2.3}{3.3}\right)^i\left(\frac{1}{3.3}\right)^{10} f_S(n-i)$
$f_X(0) = \frac{1}{3.3^{10}}$

$$f_S(0) = e^{-2.4}\sum_{n=0}^{\infty}\frac{2.4^n}{3.3^{10n}n!} = e^{\frac{2.4}{3.3^{10}}-2.4} = 0.09071937$$

$$f_S(1) = \frac{2.4}{3.3^{10}}\left(10\times\frac{2.3}{3.3}\times f_S(0)\right) = 0.000009907995$$

$$f_S(2) = \frac{2.4}{3.3^{10}}\left(\frac{10}{2}\times\frac{2.3}{3.3}\times f_S(1) + \frac{10\times 11}{2}\times\left(\frac{2.3}{3.3}\right)^2\times f_S(0)\right) = 0.00003798119$$

$$f_S(3) = \frac{2.4}{3.3^{10}}\left(\frac{10}{3}\times\frac{2.3}{3.3}\times f_S(2) + \frac{2\times 10\times 11}{3\times 2}\times\left(\frac{2.3}{3.3}\right)^2\times f_S(1) + \frac{10\times 11\times 12}{6}\times\left(\frac{2.3}{3.3}\right)^3\times f_S(0)\right) = 0.0001058901$$

So the probability that the aggregate loss is at most 3 is therefore $0.09071937+0.000009907995+0.00003798119+0.0001058901 = 0.09087315$

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For the zero-truncated ETNB distribution, we have that
$a = \frac{\beta}{1+\beta} = \frac{3}{4}$, $b = \frac{(r-1)\beta}{1+\beta} = -1.2$

$$
\begin{aligned}
q_1 &= \frac{-0.6 \times 3}{4(4^{-0.6} - 1)} = 0.7968484 \\
q_2 &= \left(\frac{3}{4} - \frac{1.2}{2}\right) 0.7968484 = 0.119527260 \\
q_3 &= \left(\frac{3}{4} - \frac{1.2}{3}\right) 0.119527260 = 0.04183454100 \\
q_4 &= \left(\frac{3}{4} - \frac{1.2}{4}\right) 0.04183454100 = 0.018825543450 \\
q_5 &= \left(\frac{3}{4} - \frac{1.2}{5}\right) 0.018825543450 = 0.009601027159 \\
q_6 &= \left(\frac{3}{4} - \frac{1.2}{6}\right) 0.009601027159 = 0.005280564937 \\
q_7 &= \left(\frac{3}{4} - \frac{1.2}{7}\right) 0.005280564937 = 0.003055183999 \\
q_8 &= \left(\frac{3}{4} - \frac{1.2}{8}\right) 0.003055183999 = 0.001833110399 \\
q_9 &= \left(\frac{3}{4} - \frac{1.2}{9}\right) 0.001833110399 = 0.0011304180793 \\
q_{10} &= \left(\frac{3}{4} - \frac{1.2}{10}\right) 0.0011304180793 = 0.0007121633899 \\
q_{11} &= \left(\frac{3}{4} - \frac{1.2}{11}\right) 0.0007121633899 = 0.0004564319907995 \\
q_{12} &= \left(\frac{3}{4} - \frac{1.2}{12}\right) 0.0004564319907995 = 0.0002966807940196 \\
q_{13} &= \left(\frac{3}{4} - \frac{1.2}{13}\right) 0.0002966807940196 = 0.0001951246760669
\end{aligned}
$$

With the deductible set at 10, the probability that a loss does not lead to a claim is $0.7968484 + 0.119527260 + 0.04183454100 + 0.018825543450 + 0.009601027159 + 0.005280564937 + 0.003055183999 + 0.001833110399 + 0.0011304180793 = 0.9981311736993669$ The distribution of the claim value is therefore

$$
\begin{aligned}
q_0 &= 0.9981311736993669 \\
q_1 &= 0.0004564319907995 \\
q_2 &= 0.0002966807940196 \\
q_3 &= 0.0001951246760669
\end{aligned}
$$

For the primary distribution $a = \frac{\beta}{1+\beta} = \frac{3}{8}$ and $b = \frac{(r-1)\beta}{1+\beta} = -0.3$.
Now we can use the recursive formula

$$f_S(n) = \frac{\sum_{i=1}^{n} \left(0.375 - \frac{0.3i}{n}\right) q_i f_S(n-i)}{1 - \frac{3}{8} \times 0.99813}$$

We calculate

$$\begin{aligned} f_S(0) &= \sum_{n=0}^{\infty} p_n (f_X(0))^n = \sum_{n=0}^{\infty} \binom{n-0.8}{n} \left(\frac{3}{8}\right)^n \left(\frac{5}{8}\right)^{0.2} (f_X(0))^n \\ &= \left(\frac{5}{8}\right)^{0.2} \left(1 - \frac{3}{8} f_X(0)\right)^{-0.2} = 0.9997759 \end{aligned}$$

Now using the recurrence, we get

$$\begin{aligned} f_S(1) &= \frac{0.075 \times 0.000456 \times 0.9998}{0.6257008} = 0.00005469823 \\ f_S(2) &= \frac{0.225 \times 0.000456 \times 0.0000547 + 0.075 \times 0.000297 \times 0.9998}{0.6257008} = 0.00003556283 \\ f_S(3) &= \frac{0.275 \times 0.000456 \times 0.0000356 + 0.175 \times 0.000297 \times 0.0000547 + 0.075 \times 0.000195 \times 0.9998}{0.6257008} = 0.00002339517 \end{aligned}$$

The probability of paying out at least \$400 to a single driver is therefore $1 - 0.9997759 - 0.00005469823 - 0.00003556283 - 0.00002339517 = 0.0001104438$

9.6.2 Applications to Compound Frequency Models

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The zero-truncated logarithmic distribution has $a = 0.8$, $b = -0.8$. This gives $p_1\left(\sum_{n=0}^{\infty}\frac{0.8^n}{n+1}\right) = 1$ $p_1 = -\frac{0.8}{\log(1-0.8)} = 0.4970679$.

$$\begin{aligned}
p_1 &= 0.4970679\\
p_2 &= 0.1988272\\
p_3 &= 0.1060412
\end{aligned}$$

Now we compound with a Poisson with $\lambda = 0.1$. The probability of the total being 0 is $e^{-0.1}$ (since the secondary distribution is zero-truncated). The recurrence is

$$f_S(n) = \sum_{i=1}^{n}\frac{0.1i}{n}p_i f_S(n-i)$$

So we calculate:

$$\begin{aligned}
f_S(1) &= 0.1 \times 0.497 \times e^{-0.1} = 0.04970679e^{-0.1}\\
f_S(2) &= 0.05 \times 0.497 \times 0.04970679e^{-0.1} + 0.1 \times 0.199e^{-0.1} = 0.0211181e^{-0.1}\\
f_S(3) &= (0.0333 \times 0.497 \times 0.0211 + 0.0667 \times 0.199 \times 0.0497 + 0.1 \times 0.106)\, e^{-0.1} = 0.010705e^{-0.1}
\end{aligned}$$

Now for the overall compound distribution, we have $f_A(0) = e^{-6}\sum_{n=0}^{\infty}\frac{6^n}{n!}e^{-0.1n} = e^{6e^{-0.1}-6} = e^{-0.5709755} =$ 0.564974.

The recurrence is

$$f_A(n) = \sum_{i=1}^{n}\frac{6i}{n}f_S(i)f_A(n-i)$$

So we calculate:

$$\begin{aligned}
f_A(1) &= 6 \times 0.0497e^{-0.1} \times 0.564974 = 0.1524635\\
f_A(2) &= 3 \times 0.04970679e^{-0.1} \times 0.1524635 + 6 \times 0.0211181e^{-0.1} \times 0.564974 = 0.08534651\\
f_A(3) &= 2 \times 0.04970679e^{-0.1} \times 0.08534651 + 4 \times 0.0211181e^{-0.1} \times 0.1524635 + 6 \times 0.010705e^{-0.1} \times 0.564974 = 0.05216554
\end{aligned}$$

So the probability that the total claimed is more than 3000 is

$$1 - 0.564974 - 0.1524635 - 0.08534651 - 0.05216554 = 0.1450505$$

9.6.2 Underflow Problems

40

The recurrence is

$$f_S(n) = \sum_{i=1}^{n} \frac{\lambda i}{n} \binom{n+3}{n} 0.6875^i 0.3125^4 f_S(n-i)$$

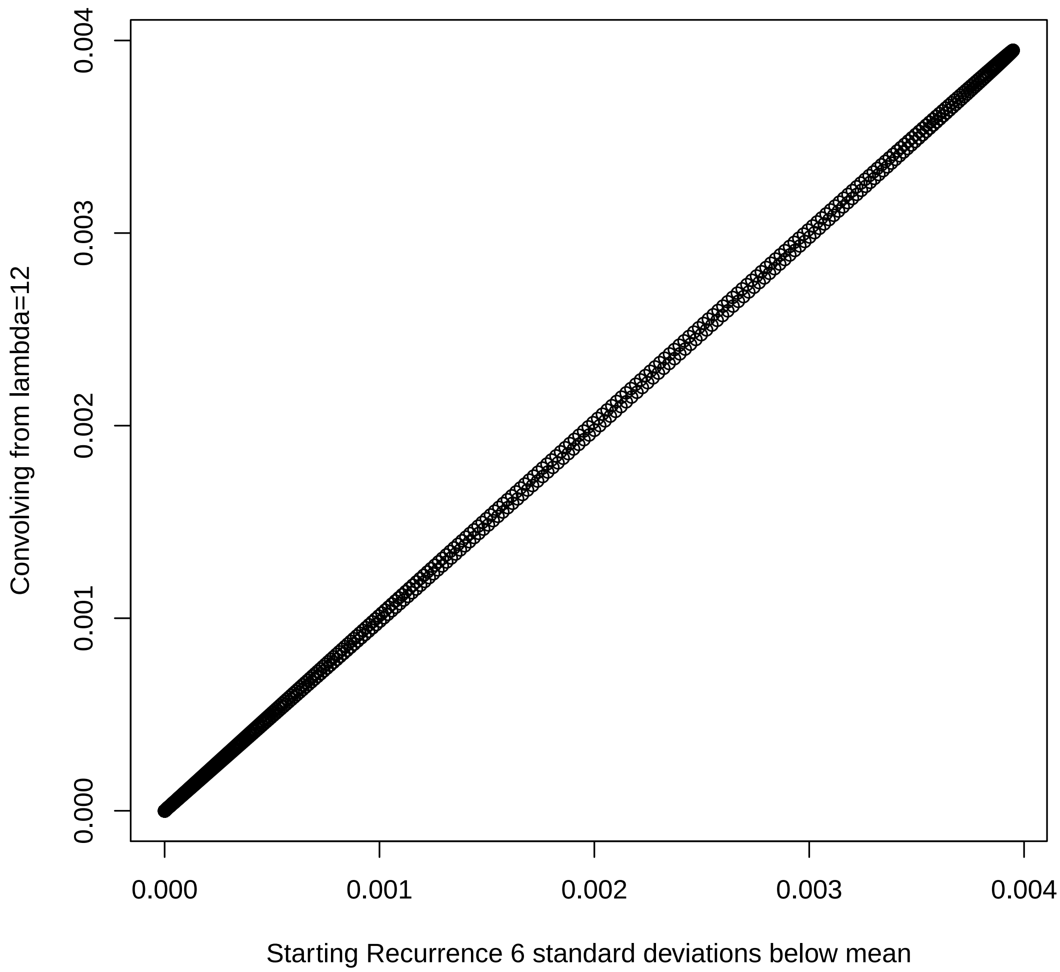
(a)

The mean of the distribution is $96 \times 8.8 = 844.8$, and the variance is $96 \times 28.16 + 96 \times 8.8^2 = 48 \times (28.16 + 77.44) = 96 \times 105.6 = 10137.6$ The standard deviation is therefore $\sqrt{10137.6} = 100.6856$, so six standard deviations below the mean is $422.4 - 6 \times 100.6856 = 240.6864$. We will start the recurrence at 241. If we assume that $f_s(240) = 0$ and $f_2(241) = 1$, then we can calculate the values

$$f_S(n) = \sum_{i=1}^{n} \frac{96i}{n} \frac{(i+1)(i+2)(i+3)}{6} 0.6875^i 0.3125^4 f_S(n-i)$$

(b) Solution for $\lambda = 12$:

$$f_S(n) = \sum_{i=1}^{n} \frac{12i}{n} \frac{(i+1)(i+2)(i+3)}{6} 0.6875^i 0.3125^4 f_S(n-i)$$



0.000
0.001
0.002
0.003
0.004
Convolving from lambda=12
0.000
0.001
0.002
0.003
0.004
Starting Recurrence 6 standard deviations below mean

9.6.3 Numerical Stability

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For the binomial distribution $a = -\frac{p}{1-p}$ $b = (n+1)\frac{p}{1-p}$
The recurrence relation is

$$\begin{aligned} f_S(x) &= \frac{1}{1 - af_X(0)} \left(\sum_{y=1}^{x} \left(a + b\frac{y}{x} \right) f_x(y) f_s(x-y) \right) \\ &= \frac{1}{1 + \frac{p}{1-p} f_X(0)} \frac{p}{1-p} \left(\sum_{y=1}^{x} \left((n+1)\frac{y}{x} - 1 \right) f_x(y) f_s(x-y) \right) \\ &= \frac{p}{1 - p + pf_X(0)} \left(\sum_{y=1}^{x} \left((n+1)\frac{y}{x} - 1 \right) f_x(y) f_s(x-y) \right) \end{aligned}$$

Substituting $p = 0.8$, $n = 7$ and $f_x(0) = 0.21$, $f_x(1) = 0.41$ and $f_x(3) = 0.38$, this becomes

$$\begin{aligned} f_S(x) &= \frac{0.8}{1 - 0.8 + 0.8 \times 0.21} \left(0.41 \left(\frac{n+1}{x} - 1 \right) f_s(x-1) + 0.38 \left(3\frac{n+1}{x} - 1 \right) f_s(x-3) \right) \\ &= \frac{50}{23} \left(0.41 \left(\frac{n+1}{x} - 1 \right) f_s(x-1) + 0.38 \left(3\frac{n+1}{x} - 1 \right) f_s(x-3) \right) \\ &= \left(\frac{41}{46} \left(\frac{n+1}{x} - 1 \right) f_s(x-1) + \frac{38}{46} \left(3\frac{n+1}{x} - 1 \right) f_s(x-3) \right) \end{aligned}$$

9.6.5 Constructing Arithmetic Distributions

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(a) Using the method of rounding, we have
$p_0 = 1 - e^{-\frac{1}{2\theta}} = 1 - e^{-\frac{1}{4}}$ $p_n = e^{-\frac{2n-1}{2\theta}} \left((1 - e^{-\frac{1}{\theta}}\right) = e^{-\frac{2n-1}{4}} \left((1 - e^{-\frac{1}{2}}\right)$
This is a zero modified geometric distribution.
(b) On the interval $[a, a+2]$, we have

$$\frac{1}{2}\int_a^{a+2} e^{-\frac{x}{2}}\,dx = e^{-\frac{a}{2}}(1-e^{-\frac{2}{2}}) = 0.632120558829e^{-\frac{a}{2}}$$

$$\begin{aligned}\frac{1}{2}\int_a^{a+2} xe^{-\frac{x}{2}}\,dx &= e^{-\frac{a}{2}}\left(a(1-e^{-\frac{2}{2}}) + \int_0^2 \frac{t}{2}e^{-\frac{t}{2}}\,dt\right)\\ &= e^{-\frac{a}{2}}\left(a(1-e^{-\frac{2}{2}}) + \left[-te^{-\frac{t}{2}}\right]_0^2 + \int_0^2 e^{-\frac{t}{2}}\,dt\right)\\ &= e^{-\frac{a}{2}}\left(a(1-e^{-1}) - 2e^{-1} + 2\left(1-e^{-1}\right)\right)\\ &= e^{-\frac{a}{2}}\left(0.632120558829a + 0.52848223532\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\int_a^{a+2} x^2e^{-\frac{x}{2}}\,dx &= e^{-\frac{a}{2}}\int_0^2 (a^2+2at+t^2)\frac{e^{-\frac{t}{2}}}{2}\,dt = e^{-\frac{a}{2}}\left(a^2\int_0^2 \frac{e^{-\frac{t}{2}}}{2}\,dt + 2a\int_0^2 \frac{te^{-\frac{t}{2}}}{2}\,dt + \int_0^2 \frac{t^2e^{-\frac{t}{2}}}{2}\,dt)\right)\\ &= e^{-\frac{a}{2}}\left(0.632120558829a^2 + 2\times 0.52848223532a + \int_0^2 t^2\frac{e^{-\frac{t}{2}}}{2}\,dt\right)\\ &= e^{-\frac{a}{2}}\left(0.632120558829a^2 + 1.05696447064a + \left[-t^2e^{-\frac{t}{2}}\right]_0^2 + 4\int_0^2 \frac{te^{-\frac{t}{2}}}{2}\,dt\right)\\ &= e^{-\frac{a}{2}}\left(0.632120558829a^2 + 1.05696447064a - 4e^{-1} + 4\times 0.52848223532\right)\\ &= e^{-\frac{a}{2}}\left(0.632120558829a^2 + 1.05696447064a + 0.6424111766\right)\end{aligned}$$

Therefore, matching moments on this interval gives us

$$\begin{aligned}p_a + p_{a+1} + p_{a+2} &= 0.632120558829e^{-\frac{a}{2}}\\ ap_a + (a+1)p_{a+1} + (a+2)p_{a+2} &= \left(0.632120558829a + 0.52848223532\right)e^{-\frac{a}{2}}\\ a^2p_a + (a+1)^2p_{a+1} + (a+2)^2p_{a+2} &= \left(0.632120558829a^2 + 1.05696447063a + 0.64241117658\right)e^{-\frac{a}{2}}\end{aligned}$$

We solve these:

$$
\begin{aligned}
p_{a+1} + 2p_{a+2} &= 0.52848223532e^{-\frac{a}{2}} \\
(a+1)p_{a+1} + 2(a+2)p_{a+2} &= (0.52848223532a + 0.64241117658)\, e^{-\frac{a}{2}} \\
2p_{a+2} &= 0.11392894126e^{-\frac{a}{2}} \\
p_{a+2} &= 0.05696447063e^{-\frac{a}{2}} \\
p_{a+1} &= 0.41455329406e^{-\frac{a}{2}} \\
p_a &= 0.160602794139e^{-\frac{a}{2}}
\end{aligned}
$$

Thus for an odd number

$$p_{2n+1} = 0.41455329406e^{-n}$$

and for an even number

$$
\begin{aligned}
p_{2n} &= 0.160602794139e^{-n} + 0.05696447063e^{-(n-1)} \\
&= 0.31544827952e^{-n}
\end{aligned}
$$

16 Model Selection

16.3 Graphical Comparison

43

The log-likelihood of this Pareto distribution is

$$14\left(\log(\alpha)+\alpha\log(\theta)\right)-(\alpha+1)\left(\log(\theta+325)+\log(\theta+692)+\log(\theta+1340)+\log(\theta+1784)+\log(\theta+1920)+\log(\theta+2503)+\log(\theta+\right.$$

Differentiating with respect to α and θ give

$$\frac{14}{\alpha}=\left(\log(\theta+325)+\log(\theta+692)+\log(\theta+1340)+\log(\theta+1784)+\log(\theta+1920)+\log(\theta+2503)+\log(\theta+3238)+\log(\theta+4054)\right.$$

$$\frac{\alpha}{\theta}=(\alpha+1)\left(\frac{1}{\theta+325}+\frac{1}{\theta+692}+\frac{1}{\theta+1340}+\frac{1}{\theta+1784}+\frac{1}{\theta+1920}+\frac{1}{\theta+2503}+\frac{1}{\theta+3238}+\frac{1}{\theta+4054}+\frac{1}{\theta+5862}+\frac{1}{\theta+6304}+\right.$$

$\theta=4156615$ $\alpha=934.25$

44
See next slide.

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$1 - e^{-\frac{10}{5.609949}} = 0.8317909$

50

$1 - e^{-\frac{3}{5.609949}} = 0.4141926$

51
$0.6^{0.8725} = 0.6403788$

16.4 Hypothesis Tests

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(a) $1-\left(\frac{4156615}{4156615+5862}\right)^{934.25}=0.7319624$ $1-\left(\frac{4156615}{4156615+9984}\right)^{934.25}=0.8936835$
$D=0.1605338$

- At the 95% level, the critical value is $\frac{1.36}{\sqrt{14}}=0.3634753$.
- At the 95% level, the critical value is $\frac{1.22}{\sqrt{14}}=0.3260587$.

so we cannot reject the model.
(b) We have that $F(x)=1-\frac{\theta^\alpha}{(x+\theta)^\alpha}$, so the statistic is

$$n\int_t^u \frac{\left(F_n(x)+\frac{\theta^\alpha}{(x+\theta)^\alpha}-1\right)^2}{\left(\frac{\theta^\alpha((x+\theta)^\alpha-\theta^\alpha)}{(x+\theta)^{2\alpha}}\right)}\left(\frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}\right)dx$$

$$n\int_t^u \alpha(x+\theta)^{\alpha-1}\frac{\left(F_n(x)+\frac{\theta^\alpha}{(x+\theta)^\alpha}-1\right)^2}{((x+\theta)^\alpha-\theta^\alpha)}dx$$

$$n\int_t^u \alpha\frac{(F_n(x)(x+\theta)^\alpha-((x+\theta)^\alpha-\theta^\alpha))^2}{(x+\theta)^{\alpha+1}((x+\theta)^\alpha-\theta^\alpha)}dx$$

$$n\int_t^u \frac{\alpha}{(x+\theta)}\left(F_n(x)^2\frac{(x+\theta)^\alpha}{((x+\theta)^\alpha-\theta^\alpha)}-2F_n(x)+\frac{((x+\theta)^\alpha-\theta^\alpha)}{(x+\theta)^\alpha}\right)dx$$

For a constant value $F_n(x)=c$, we have

$$\begin{aligned}
&\int_a^b c^2\frac{\alpha(x+\theta)^{\alpha-1}}{((x+\theta)^\alpha-\theta^\alpha)}-\frac{2\alpha c}{x+\theta}+\frac{\alpha((x+\theta)^\alpha-\theta^\alpha)}{(x+\theta)^{\alpha+1}}dx\\
&=c^2\left[\log((x+\theta)^\alpha-\theta^\alpha)\right]_a^b-2\alpha c\left[\log(x+\theta)\right]_a^b+\alpha\left[\log(x+\theta)\right]_a^b-\left[-\frac{\theta^\alpha}{(x+\theta)^\alpha}\right]_a^b\\
&=c^2\log\left(\frac{(b+\theta)^\alpha-\theta^\alpha}{(a+\theta)^\alpha-\theta^\alpha}\right)+\alpha(1-2c)\log\left(\frac{b+\theta}{a+\theta}\right)+\frac{\theta^\alpha}{(b+\theta)^\alpha}-\frac{\theta^\alpha}{(a+\theta)^\alpha}\\
&=c^2\log\left(\frac{(a+\theta)^\alpha((b+\theta)^\alpha-\theta^\alpha)}{(b+\theta)^\alpha((a+\theta)^\alpha-\theta^\alpha)}\right)+\alpha(1-c)^2\log\left(\frac{b+\theta}{a+\theta}\right)+\frac{\theta^\alpha}{(b+\theta)^\alpha}-\frac{\theta^\alpha}{(a+\theta)^\alpha}
\end{aligned}$$

For our example, if we let $t=0$ and $u=\infty$, we have the following:

$$
\begin{aligned}
&14\Bigg(\alpha\Bigg(1^2\log\left(\frac{325+\theta}{\theta}\right)+\left(\frac{13}{14}\right)^2\log\left(\frac{692+\theta}{325+\theta}\right)+\left(\frac{12}{14}\right)^2\log\left(\frac{1340+\theta}{692+\theta}\right)+\left(\frac{11}{14}\right)^2\log\left(\frac{1784+\theta}{1340+\theta}\right)\\
&+\left(\frac{10}{14}\right)^2\log\left(\frac{1920+\theta}{1784+\theta}\right)+\left(\frac{9}{14}\right)^2\log\left(\frac{2503+\theta}{1920+\theta}\right)+\left(\frac{8}{14}\right)^2\log\left(\frac{3238+\theta}{2503+\theta}\right)+\left(\frac{7}{14}\right)^2\log\left(\frac{4054+\theta}{3238+\theta}\right)\\
&+\left(\frac{6}{14}\right)^2\log\left(\frac{5862+\theta}{4054+\theta}\right)+\left(\frac{6}{14}\right)^2\log\left(\frac{6304+\theta}{5862+\theta}\right)+\left(\frac{5}{14}\right)^2\log\left(\frac{6926+\theta}{6304+\theta}\right)+\left(\frac{4}{14}\right)^2\log\left(\frac{6926+\theta}{6304+\theta}\right)\\
&+\left(\frac{3}{14}\right)^2\log\left(\frac{8120+\theta}{6926+\theta}\right)+\left(\frac{2}{14}\right)^2\log\left(\frac{9176+\theta}{8120+\theta}\right)+\left(\frac{1}{14}\right)^2\log\left(\frac{9984+\theta}{9176+\theta}\right)\Bigg)\\
&+\left(\frac{1}{14}\right)^2\log\left(\frac{1-\left(\frac{\theta}{692+\theta}\right)^\alpha}{1-\left(\frac{\theta}{325+\theta}\right)^\alpha}\right)+\cdots+\left(\frac{14}{14}\right)^2\log\left(\frac{1}{1-\left(\frac{\theta}{9984+\theta}\right)^\alpha}\right)-1\Bigg)\\
&=0.3873562
\end{aligned}
$$

So the model cannot be rejected.

54
If the parameter of the Exponential distribution is θ, then the log-likelihood of the data is

$$742\log(1-e^{-\left(\frac{5000}{\theta}\right)})+1304\log(e^{-\left(\frac{5000}{\theta}\right)}-e^{-\left(\frac{10000}{\theta}\right)})+1022\log(e^{-\left(\frac{10000}{\theta}\right)}-e^{-\left(\frac{15000}{\theta}\right)})+ \\ 830\log(e^{-\left(\frac{15000}{\theta}\right)}-e^{-\left(\frac{20000}{\theta}\right)})+211\log(e^{-\left(\frac{20000}{\theta}\right)}-e^{-\left(\frac{25000}{\theta}\right)})-143\left(\frac{25000}{\theta}\right)$$

Taking the derivative with respect to θ, we get

$$742\frac{5000e^{-\left(\frac{5000}{\theta}\right)}}{\theta^2(1-e^{-\left(\frac{5000}{\theta}\right)})}+1304\frac{(5000e^{-\left(\frac{5000}{\theta}\right)}-10000e^{-\left(\frac{10000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{5000}{\theta}\right)}-e^{-\left(\frac{10000}{\theta}\right)})}+ \\ 1022\frac{(10000e^{-\left(\frac{10000}{\theta}\right)}-15000e^{-\left(\frac{15000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{10000}{\theta}\right)}-e^{-\left(\frac{15000}{\theta}\right)})}+830\frac{(15000e^{-\left(\frac{10000}{\theta}\right)}-20000e^{-\left(\frac{20000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{15000}{\theta}\right)}-e^{-\left(\frac{20000}{\theta}\right)})}+ \\ 211\frac{(20000e^{-\left(\frac{20000}{\theta}\right)}-25000e^{-\left(\frac{25000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{20000}{\theta}\right)}-e^{-\left(\frac{25000}{\theta}\right)})}-143\frac{25000}{\theta^2}=0$$

Multiplying by $\frac{\theta^2(1-e^{-\frac{5000}{\theta}})}{5000}$ gives

$$\begin{aligned}742e^{-\left(\frac{5000}{\theta}\right)}+1304(1-2e^{-\left(\frac{5000}{\theta}\right)})+1022(2-3e^{-\left(\frac{5000}{\theta}\right)})+830(3-4e^{-\left(\frac{5000}{\theta}\right)})+& \\ 211(4-5e^{-\left(\frac{5000}{\theta}\right)})-143(5-5e^{-\left(\frac{5000}{\theta}\right)})&=0 \\ 5967-10076e^{-\left(\frac{5000}{\theta}\right)}&=0 \\ e^{-\left(\frac{5000}{\theta}\right)}&=\frac{5967}{10076} \\ \theta&=\frac{5000}{\log\left(\frac{10076}{5967}\right)} \\ &=9543.586\end{aligned}$$

This gives the following table

Claim Amount	O_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0–5,000	742	1733.969	567.49
5,000–10,000	1304	1026.855	74.80
10,000–15,000	1022	608.103	281.71
15,000–20,000	830	360.118	613.10
20,000–25,000	211	213.262	0.02
More than 25,000	143	309.694	89.72
total			1626.85

This should be compared to a Chi-square with 5 degrees of freedom, so the model is rejected at all significance levels.

55

For the exponential distribution, the log-likelihood is

$$-\left(\frac{382}{\theta}+\frac{596}{\theta}+\frac{920}{\theta}+\frac{1241}{\theta}+\frac{1358}{\theta}+\frac{1822}{\theta}+\frac{2010}{\theta}+\frac{2417}{\theta}+\frac{2773}{\theta}+\frac{3002}{\theta}+\frac{3631}{\theta}+\frac{4120}{\theta}+\frac{4692}{\theta}+\frac{5123}{\theta}+14\log(\theta)\right)$$

This is maximised by

$$\theta=\frac{382+596+920+1241+1358+1822+2010+2417+2773+3002+3631+4120+4692+5123}{14}=2434.786$$

Which gives a log-likelihood of $-(14+14\log(2434.786))=-123.1666$.

For the Weibull distribution, the log-likelihood is

$$14\log(\tau)+(\tau-1)(\log(382)+\cdots+\log(5213))-\left(\left(\frac{382}{\theta}\right)^{\tau}+\cdots+\left(\frac{5123}{\theta}\right)^{\tau}+14\tau\log(\theta)\right)$$

Setting the derivatives with respect to θ and τ equal to zero gives:

$$\begin{aligned}\tau\left(\frac{382^{\tau}}{\theta^{\tau+1}}+\cdots+\frac{5123^{\tau}}{\theta^{\tau+1}}-\frac{14}{\theta}\right)&=0\\ \frac{382^{\tau}+\cdots+5123^{\tau}}{14}&=\theta^{\tau}\\ \frac{14}{\tau}+(\log(382)+\cdots+\log(5213))-\left(\left(\frac{382}{\theta}\right)^{\tau}\log\left(\frac{382}{\theta}\right)+\cdots+\left(\frac{5123}{\theta}\right)^{\tau}\log\left(\frac{5123}{\theta}\right)\right)-14\log(\theta)&=0\\ \frac{14}{\tau}+\left(1-\left(\frac{382}{\theta}\right)^{\tau}\right)\log(382)+\cdots+\left(1-\left(\frac{5123}{\theta}\right)^{\tau}\right)\log(5123)&=0\end{aligned}$$

This gives the solution $\tau=1.695356$ and $\theta=2729.417$

$l(x;\tau,\theta)=-120.7921$

The log-likelihood ratio statistic is therefore

$$2(-120.7921-(-123.1666))=4.749$$

For a Chi-square with 1 degree of freedom, this has a p-value 0.04703955, so the Weibull model is preferred at the 5% significance level.

Study Note: Information Criteria

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For the inverse exponential distribution, with parameter θ the likelihood is $\prod_{i=1}^{n} \frac{\theta e^{-\frac{\theta}{x}}}{x^2}$ and the log-likelihood is

$$l(\theta) = n\log(\theta) + \sum_{i=1}^{n} -2\log(x) - \frac{\theta}{x}$$

$$\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x}$$

So the likelihood is maximised by $\theta = \frac{n}{\sum_{i=1}^{n} \frac{1}{x}} = 1399.291$. This gives a log-likelihood of $14\log(1399.291) - \sum_{i=1}^{n} 2\log(x) - \theta\sum_{i=1}^{n} \frac{1}{x} = -124.292$.

Now for the AIC, we get:

Weibull: $-120.7921 - 2 = -122.7921$ Inverse Exponential: $-124.292 - 1 = -125.292$

For BIC, we get

Weibull: $-120.7921 - \frac{2}{2}\log(14) = - -123.4312$ Inverse Exponential: $-124.292 - \frac{1}{2}\log(14) = -125.6115$

18 Greatest Accuracy Credibility

18.2 Conditional Distributions and Expectation

57

(a) Let $\Theta = 1$ for frequent drivers, and $\Theta = 0$ for infrequent drivers. Then

$$\begin{aligned}\mathbb{E}(X|\Theta = 1) &= 0.4\\ \mathbb{E}(X|\Theta = 0) &= 0.1\\ \mathrm{Var}(X|\Theta = 1) &= 0.4\\ \mathrm{Var}(X|\Theta = 0) &= 0.1\end{aligned}$$

so

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Theta) = 0.75 \times 0.4 + 0.25 \times 0.1 = 0.325$$

and

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\theta)) + \mathrm{Var}(\mathbb{E}(X|\Theta)) = 0.325 + 0.3^2 \times 0.25 \times 0.75 = 0.325 + 0.016875 = 0.341875$$

(b)

$$P(X = 0|\Theta) = \begin{cases} e^{-0.4} & \text{if } \Theta = 1 \\ e^{-0.1} & \text{if } \Theta = 0 \end{cases}$$

So

$$P(\Theta = 1|X = 0) = \frac{0.75e^{-0.4}}{0.75e^{-0.4} + 0.25e^{-0.1}} = 0.6896776$$

Therefore the new expectation and variance are:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Theta) = 0.6896776 \times 0.4 + 0.3103224 \times 0.1 = 0.3069033$$

and

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\theta)) + \mathrm{Var}(\mathbb{E}(X|\Theta)) = 0.3069033 + 0.3^2 \times 0.3103224 \times 0.6896776 = 0.325 + 0.016875 = 0.3261653$$

18.3 Bayesian Methodology

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(a) We have $\mathbb{E}(X|\Theta=\theta)=\frac{\theta}{2}$, so

$$\mathbb{E}(X)=\mathbb{E}(\mathbb{E}(X|\Theta=\theta))=\mathbb{E}\left(\frac{\Theta}{2}\right)=150$$

(b) The joint density function is

$$f_{X,\Theta}(x,\theta)=\left(\frac{\theta^2}{2\times 100^3}e^{-\frac{\theta}{100}}\right)\left(\frac{\theta^3}{2x^4}e^{-\frac{\theta}{x_1}}\right)$$

For samples, x_1 and x_2, the joint density is therefore

$$\begin{aligned}\left(\frac{\theta^2}{2000000}e^{-\frac{\theta}{100}}\right)\left(\frac{\theta^3}{2{x_1}^4}e^{-\frac{\theta}{x_1}}\right)\left(\frac{\theta^3}{2{x_2}^4}e^{-\frac{\theta}{x_2}}\right)\\ =\frac{\theta^8}{8000000{x_1}^4{x_2}^4}e^{-\theta\left(\frac{1}{100}+\frac{1}{x_1}+\frac{1}{x_2}\right)}\end{aligned}$$

The posterior distribution of Θ is therefore a gamma distribution with $\alpha=9$ and $\theta=\frac{1}{\frac{1}{100}+\frac{1}{x_1}+\frac{1}{x_2}}=$ 43.29897.

The expected aggregate losses are given by

$$\begin{aligned}\mathbb{E}(X)=\mathbb{E}(\mathbb{E}(X|\Theta))&=\frac{\mathbb{E}(\Theta)}{2}\\ &=4.5\times 43.29897\\ &=194.845365\end{aligned}$$

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(a) We have

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Lambda)) = \mathbb{E}\left(\Lambda\right) = 1$$

(b) The posterior distribution is a Gamma distribution with $\alpha = 0.5 + m$ and $\theta = \frac{2}{1+2n}$. We therefore have

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|\Lambda)) = \mathbb{E}\left(\Lambda\right) \\
&= \frac{2(m+0.5)}{1+2n} \\
&= \frac{2m+1}{2n+1} \\
&= \left(\frac{2n}{1+2n}\right)\left(\frac{m}{n}\right) + \left(\frac{1}{1+2n}\right)
\end{aligned}$$

60

The posterior density function is proportional to

$$4\lambda^m e^{-n\lambda}\frac{3^4}{(3+\lambda)^5}$$

We have that the posterior expected number of claims is the posterior expected value of Λ, which is given by

$$\frac{\int_0^\infty \frac{\lambda^{m+1}e^{-n\lambda}}{(3+\lambda)^5}\,d\lambda}{\int_0^\infty \frac{\lambda^{m}e^{-n\lambda}}{(3+\lambda)^5}\,d\lambda}$$

Substituting $u=\lambda+3$, these integrals become

$$\frac{\int_3^\infty u^{-5}(u-3)^{m+1}e^{-nu}\,du}{\int_3^\infty u^{-5}(u-3)^{m}e^{-nu}\,du}$$

The credibility estimate is then given in the following table:

	1	2	3	4	5	6	7	8	9	10
0	0.4331	0.2937	0.2243	0.1821	0.1534	0.1327	0.1170	0.1046	0.0946	0.0864
1	0.9261	0.6073	0.4580	0.3693	0.3101	0.2675	0.2354	0.2103	0.1900	0.1734
2	1.4785	0.9396	0.7003	0.5614	0.4697	0.4044	0.3552	0.3169	0.2862	0.2609
3	2.0874	1.2891	0.9506	0.7579	0.6321	0.5430	0.4764	0.4246	0.3831	0.3491
4	2.7487	1.6543	1.2081	0.9584	0.7970	0.6833	0.5987	0.5331	0.4806	0.4377
5	3.4571	2.0336	1.4722	1.1627	0.9642	0.8252	0.7221	0.6424	0.5788	0.5269
6	4.2067	2.4256	1.7423	1.3704	1.1336	0.9686	0.8466	0.7525	0.6776	0.6165
7	4.9919	2.8288	2.0178	1.5811	1.3050	1.1134	0.9721	0.8633	0.7769	0.7065
8	5.8073	3.2420	2.2981	1.7948	1.4782	1.2594	1.0985	0.9749	0.8768	0.7970
9	6.6477	3.6640	2.5829	2.0110	1.6531	1.4065	1.2257	1.0870	0.9771	0.8878

We compare this to the table of $\frac{2m+1}{2n+1}$ that we get from the Gamma prior.

	1	2	3	4	5	6	7	8	9	10
0	0.3333	0.2000	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476
1	1.0000	0.6000	0.4286	0.3333	0.2727	0.2308	0.2000	0.1765	0.1579	0.1429
2	1.6667	1.0000	0.7143	0.5556	0.4545	0.3846	0.3333	0.2941	0.2632	0.2381
3	2.3333	1.4000	1.0000	0.7778	0.6364	0.5385	0.4667	0.4118	0.3684	0.3333
4	3.0000	1.8000	1.2857	1.0000	0.8182	0.6923	0.6000	0.5294	0.4737	0.4286
5	3.6667	2.2000	1.5714	1.2222	1.0000	0.8462	0.7333	0.6471	0.5789	0.5238
6	4.3333	2.6000	1.8571	1.4444	1.1818	1.0000	0.8667	0.7647	0.6842	0.6190
7	5.0000	3.0000	2.1429	1.6667	1.3636	1.1538	1.0000	0.8824	0.7895	0.7143
8	5.6667	3.4000	2.4286	1.8889	1.5455	1.3077	1.1333	1.0000	0.8947	0.8095
9	6.3333	3.8000	2.7143	2.1111	1.7273	1.4615	1.2667	1.1176	1.0000	0.9048

18.4 The Credibility Premium

61

We are trying to choose α_i to minimise

$$\mathbb{E}\left(\mu(\Theta)-\left(\alpha_0+\sum_{i=1}^{n}\alpha_i X_i\right)\right)^2=\mathbb{E}\left(\mu(\Theta)^2-2\mu(\Theta)\left(\alpha_0+\sum_{i=1}^{n}\alpha_i X_i\right)+\left(\alpha_0+\sum_{i=1}^{n}\alpha_i X_i\right)^2\right)$$
$$=\mathbb{E}\left(\mu(\Theta)^2\right)-2\alpha_0\mathbb{E}\mu(\Theta)-\mathbb{E}\left(\mu(\Theta)\left(\sum_{i=1}^{n}\alpha_i X_i\right)\right)+{\alpha_0}^2+2\alpha_0\mathbb{E}\left(\sum_{i=1}^{n}\alpha_i X_i\right)+\mathbb{E}\left(\left(\sum_{i=1}^{n}\alpha_i X_i\right)^2\right)$$
$$=\mu^2+v^2-2\alpha_0\mu+{\alpha_0}^2+2\alpha_0\mathbb{E}\left(\sum_{i=1}^{n}\alpha_i X_i\right)-\mathbb{E}\left(\mu(\Theta)\left(\sum_{i=1}^{n}\alpha_i X_i\right)\right)+\mathbb{E}\left(\left(\sum_{i=1}^{n}\alpha_i X_i\right)^2\right)$$

Setting the derivative with respect to α_0 equal to zero yields

$$2\left(\alpha_0+\mathbb{E}\left(\sum_{i=1}^{n}\alpha_i X_i\right)-\mu\right)=0$$

That is, α_0 should be chosen to make the estimate unbiassed. Now we differentiate with respect to α_j, and set the derivative equal to zero:

$$2\left(\alpha_0\mathbb{E}(X_j)-\mathbb{E}(\mu(\Theta)X_j)+\mathbb{E}\left(X_j\sum_{i=1}^{n}\alpha_i X_i\right)\right)=0$$
$$2\left(\mathbb{E}\left(\mu(\Theta)-\sum_{i=1}^{n}\alpha_i X_i\right)\mathbb{E}(X_j)-\mathbb{E}\left(X_j\left(\mu(\Theta)-\sum_{i=1}^{n}\alpha_i X_i\right)\right)\right)=0$$
$$\mathrm{Cov}\left(X_j,\mu(\Theta)-\sum_{i=1}^{n}\alpha_i X_i\right)=0$$
$$\mathrm{Cov}\left(X_j,\mu(\Theta)\right)=\sum_{i=1}^{n}\alpha_i\,\mathrm{Cov}\left(X_j,X_i\right)$$

Since X_i and X_{n+1} are conditionally independent given $\mu(\Theta)$, we have that $\mathrm{Cov}\left(X_j,\mu(\Theta)\right)=\mathrm{Cov}\left(X_j,X_{n+1}\right)$

In this situation, the second normal equations becomes:

$$\rho = \left(\sum_{i=1}^{n} \alpha_i \rho\right) + \alpha_j \sigma^2$$
$$\alpha_j = \frac{\rho\left(1 - \sum_{i=1}^{n} \alpha_i\right)}{\sigma^2}$$

So all the α_j are equal to a common value α, and we get

$$\alpha = \frac{\rho\left(1 - n\alpha\right)}{\sigma^2}$$

Now we have $\mathbb{E}(X_{n+1}) = \mu = \mathbb{E}(X_i)$ The first normal equation then becomes

$$\begin{aligned} \mu &= \alpha_0 + \left(\sum_{i=1}^{n} \alpha_i\right) \mu \\ &= \alpha_0 + n\alpha\mu \\ \alpha_0 &= (1 - n\alpha)\mu \end{aligned}$$

We can therefore rewrite our credibility estimate as

$$Z\overline{X} + (1 - Z)\mu$$

where $Z = n\alpha$. We can then solve:

$$\begin{aligned} \frac{Z}{n} &= \frac{\rho\left(1 - Z\right)}{\sigma^2} \\ \sigma^2 Z &= n\rho(1 - Z) \\ (\sigma^2 + n\rho)Z &= n\rho \\ Z &= \frac{n}{n + \frac{\sigma^2}{\rho}} \end{aligned}$$

63

Let the coefficients of the X_i be α_i, and let the coefficients of Y_i be β_i. The normal equations are:

$$\begin{aligned}
\mu + \nu &= \alpha_0 + \sum_{j=1}^{n} \alpha_i \mu + \sum_{k=1}^{m} \beta_i \nu \\
\rho + \xi &= \sum_{j \neq i} \alpha_j \rho + \sum_{k=1}^{m} \beta_k \xi + \alpha_i \sigma^2 \\
\zeta + \xi &= \sum_{j=1}^{n} \alpha_j \xi + \sum_{k \neq i} \beta_k \zeta + \beta_i \tau^2
\end{aligned}$$

From these, we deduce that $\beta_i(\tau - \zeta) = \beta_j(\tau - \zeta)$, and so $\beta_i = \beta_j = \beta$ (assuming the Y_i are not perfectly correlated). Similarly, $\alpha_i = \alpha_j = \alpha$. Substituting these into the normal equations gives:

$$\begin{aligned}
\mu + \nu &= \alpha_0 + n\alpha\mu + m\beta\nu \\
\rho + \xi &= \alpha((n-1)\rho + \sigma^2) + m\beta\xi \\
\zeta + \xi &= n\alpha\xi + \beta((m-1)\zeta + \tau^2)
\end{aligned}$$

This gives

$$\begin{aligned}
\left(\frac{(n-1)\rho + \tau}{n\xi}\right)(\zeta + \xi) - (\rho + \xi) &= \left(\frac{(n-1)\rho + \sigma^2}{n\xi}\right)\beta((m-1)\zeta + \tau^2) - m\beta\xi \\
\beta &= \frac{\left(\frac{(n-1)\rho+\sigma^2}{n\xi}\right)(\zeta + \xi) - (\rho + \xi)}{\left(\frac{(n-1)\rho+\sigma^2}{n\xi}\right)((m-1)\zeta + \tau^2) - m\xi} \\
&= \frac{\left((n-1)\rho + \sigma^2\right)(\zeta + \xi) - n\xi(\rho + \xi)}{((n-1)\rho + \sigma^2)\left((m-1)\zeta + \tau^2\right) - mn\xi^2} \\
\alpha &= \frac{((m-1)\zeta + \tau^2) - m\xi(\zeta + \xi)}{((n-1)\rho + \sigma^2)\left((m-1)\zeta + \tau^2\right) - mn\xi^2} \\
\alpha_0 &= (1 - n\alpha)\mu + (1 - m\beta)\nu
\end{aligned}$$

18.5 The Buhlmann Model

64

We have $Z = \frac{851}{851+\frac{84036}{23804}} = 0.9958687$, and $\overline{X} = \frac{121336}{851} = \$142.58$ so the credibility premium is

$$0.9958687 \times 142.58 + 0.0041313 \times 326 = 143.34$$

65

We have $Z=\frac{10}{10+\frac{732403}{28822}}=0.2823961$, and $\overline{X}=\frac{3224}{10}=322.40$ so the credibility premium is

$$0.2823961\times 322.40+0.7176039\times 990=\$801.47$$

18.6 The Buhlmann-Straub Model

66

The weighted mean is $\frac{1000000}{1242} = 805.153$. The credibility is $Z = \frac{1242}{1242+\frac{81243100}{120384}} = 0.6479325$. The credibility premium is therefore

$$0.6479325 \times 805.153 + 0.4520675 \times 1243 = \$959.30$$

67

The weighted mean is $\frac{14000}{\left(\frac{49}{12}\right)} = \$3,428.57$. The credibility is $Z = \frac{\left(\frac{49}{12}\right)}{\left(\frac{49}{12}\right)+\left(\frac{34280533}{832076}\right)} = 0.09017537$. The credibility premium is therefore

$$0.09017537 \times 3428.57 + 0.90981463 \times 600 = \$855.07$$

18.7 Exact Credibility

68

The Bayes premium is the conditional expectation of X_{n+1} given $X_1, \ldots, X_n$. We are given that it is a linear function of X_i. That is

$$\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = \alpha_0 + \sum_{i=1}^{n} \alpha_i X_i$$

Now recall that

$$\begin{aligned}
\mathrm{Cov}(X_i, X_{n+1}) &= \mathbb{E}(X_i X_{n+1}) - \mathbb{E}(X_i)\mathbb{E}(X_{n+1}) \\
&= \mathbb{E}(\mathbb{E}(X_i X_{n+1}|X_1, \ldots, X_n)) - \mathbb{E}(X_i)\mathbb{E}(\mathbb{E}(X_{n+1}|X_1, \ldots, X_n)) \\
&= \mathbb{E}(X_i \sum_{j=1}^{n} \alpha_j X_j) - \mathbb{E}(X_i)\mathbb{E}(\sum_{j=1}^{n} \alpha_j X_j) \\
&= \sum_{j=1}^{n} \alpha_j \left(\mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)\right) \\
&= \sum_{j=1}^{n} \alpha_j \, \mathrm{Cov}(X_i, X_j)
\end{aligned}$$

This means that the second normal equation is satisfied by the Bayes premium. We also have

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|X_1, \ldots, X_n)) = \mathbb{E}\left(\alpha_0 + \sum_{i=1}^{n} \alpha_i X_i\right)$$

So the first normal equation is satisfied. Thus the Bayes premium is the credibility premium. [Technically, need to show this is the only solution].

69
The conjugate prior is

$$\pi(\theta) \propto h(\theta) e^{\alpha r(\theta)} q(\theta)^{-\beta}$$

We choose $h(\theta) = Cr'(\theta)$.
We now show:

Proposition 1. *The marginal mean is $\frac{\alpha}{\beta}$.*

Proof. We have that C is given by

$$C \int_{\theta_0}^{\theta_1} r'(\theta) e^{\alpha r(\theta)} q(\theta)^{-\beta} \, d\theta = 1$$

We note that

$$r'(\theta) e^{\alpha r(\theta)} = \frac{d}{d\theta} \left(\frac{e^{\alpha r(\theta)}}{\alpha} \right)$$

so integrating by parts gives

$$C \left(\left[\frac{e^{\alpha r(\theta)}}{\alpha} q(\theta)^{-\beta} \right]_{\theta_0}^{\theta_1} + \frac{\beta}{\alpha} \int_{\theta_0}^{\theta_1} e^{\alpha r(\theta)} q'(\theta) q(\theta)^{-\beta-1} \, d\theta \right) = 1$$

We have that $\mathbb{E}(X|\Theta = \theta) = \frac{q'(\theta)}{q(\theta) r'(\theta)}$, so

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Theta)) = \mathbb{E}\left(\frac{q'(\Theta)}{q(\Theta) r'(\Theta)} \right) = \int_{\theta_0}^{\theta_1} \left(\frac{q'(\theta)}{q(\theta) r'(\theta)} \right) Cr'(\theta) e^{\alpha r(\theta)} q(\theta)^{-\beta} = C \int_{\theta_0}^{\theta_1} q'(\theta) e^{\alpha r(\theta)} q(\theta)^{-\beta-1}$$

Thus

$$C \left[\frac{e^{\alpha r(\theta)}}{\alpha} q(\theta)^{-\beta} \right]_{\theta_0}^{\theta_1} + \frac{\beta}{\alpha} \mathbb{E}(X) = 1$$

By the conditions $\frac{\pi(\theta_1)}{r'(\theta_1)} = \frac{\pi(\theta_0)}{r'(\theta_0)}$, we have that $\left[\frac{e^{\alpha r(\theta)}}{\alpha} q(\theta)^{-\beta} \right]_{\theta_0}^{\theta_1} = 0$, which completes the proof. □

The posterior distribution is

$$\frac{\pi(\theta) e^{r(\theta) \sum X_i}}{q(\theta)^N} = Cr'(\theta) e^{\alpha r(\theta) - \beta \log(q(\theta))} e^{r(\theta) \sum X_i - N \log(q(\theta))} = \frac{Cr'(\theta) e^{r(\theta) \left(\alpha + \sum_{i=1}^n X_i \right)}}{q(\theta)^{\beta + N}}$$

By the above proposition, the mean of this is $\frac{\alpha + \sum_{i=1}^n X_i}{\beta + N}$, which, for fixed N is clearly a linear function of $\frac{\sum_{i=1}^n X_i}{N}$.

19 Empirical Bayes Parameter Estimation

19.2 Nonparametric Estimation

70

(a) The overall mean is $\frac{2770.8}{8} = 346.35$

The EPV is $\frac{60595.2+1225822.8+62760.2+192962.3+0.0+30505.0+140653.7+56385.3}{8} = \frac{1769684.5}{8} = 221210.5625$

The total variance is

$\frac{(172.80-346.35)^2+(671.60-346.35)^2+(177.80-346.35)^2+(635.40-346.35)^2+(0.00-346.35)^2+(247.00-346.35)^2+(633.60-346.35)^2+(232.60-346.35)^2}{7} =$ 67592.36

The VHM is $67592.36 - \frac{221210.5625}{5} = 23350.25$

(b) The credibility of 5 years of experience is

$$Z = \frac{5}{5 + \frac{221210.5625}{23350.25}} = 0.3454569$$

The premiums are

$$\begin{aligned}
0.3454569 \times 172.80 + 0.6545431 \times 346.35 &= \$286.40 \\
0.3454569 \times 671.60 + 0.6545431 \times 346.35 &= \$458.71 \\
0.3454569 \times 177.80 + 0.6545431 \times 346.35 &= \$288.12 \\
0.3454569 \times 635.40 + 0.6545431 \times 346.35 &= \$446.20 \\
0.3454569 \times 0.00 + 0.6545431 \times 346.35 &= \$226.70 \\
0.3454569 \times 247.00 + 0.6545431 \times 346.35 &= \$312.03 \\
0.3454569 \times 633.60 + 0.6545431 \times 346.35 &= \$445.58 \\
0.3454569 \times 232.60 + 0.6545431 \times 346.35 &= \$307.05
\end{aligned}$$

71 In general, we can write the aggregate loss per unit of exposure for the ith company in the jth year as $A_{ij} = M_i + E_{ij}$ where M_i is the mean aggregate loss per unit of exposure for the ith company, and E_{ij} is the process variation, which has mean 0 and expected variance $\frac{\sigma^2}{m_{ij}}$. The estimated mean for each company is

$$A_i = \frac{1}{m_i}\sum m_{ij}A_{ij} = \frac{1}{m_i}\sum m_{ij}M_i + \frac{1}{m_i}\sum m_{ij}E_{ij} = M_i + \frac{1}{m_i}\sum m_{ij}E_{ij}$$

We let $E_i = \frac{1}{m_i}\sum m_{ij}E_{ij}$, and see that E_i has mean 0 and expected variance $\frac{\sigma^2}{m_i}$.

Now we consider

$$\sum m_i(A_i - \hat{\mu})^2 = \sum m_i \left((M_i - \overline{M_i}) + (E_i - \overline{E_i})\right)^2$$

Since the E_i are assumed to have mean 0 for each i, we should have $\mathrm{Cov}\left(M_i - \overline{M_i}, E_i - \overline{E_i}\right) = 0$. This gives

$$\mathbb{E}\left(\sum m_i(A_i - \hat{\mu})^2\right) = \sum m_i \left(\mathbb{E}((M_i - \overline{M_i})^2) + \mathbb{E}((E_i - \overline{E_i})^2)\right)$$

We have that $\frac{\sum m_i \mathbb{E}((E_i - \overline{E_i})^2)}{(n-1)} = \sigma^2$, the expected process variance. Therefore, we calculate

$$\mathbb{E}\left(\sum m_i(A_i - \hat{\mu})^2\right) - (n-1)\widehat{\sigma^2} = \sum m_i \left(\mathbb{E}((M_i - \overline{M_i})^2\right)$$

We are interested in the variance of the M_i, where the probability of each i is assumed to be $\frac{m_i}{m}$. We have that $\mathrm{Var}(M_i - \overline{M}) = \mathrm{Var}(M_i) + \mathrm{Var}(\overline{M}) - 2\,\mathrm{Cov}(M_i, \overline{M})$. We know that $\mathrm{Var}(M_i) = {\sigma_m}^2$ the variance of hypothetical means. Since $\overline{M} = \sum \frac{m_i}{m} M_i$, we have that $\mathrm{Var}(M) = {\sigma_m}^2 \sum \frac{{m_i}^2}{m^2}$ and $\mathrm{Cov}(M_i, \overline{M}) = \frac{m_i}{m}{\sigma_m}^2$. This gives that

$$\begin{aligned}\sum m_i \,\mathrm{Var}(M_i - \overline{M}) &= \sum m_i {\sigma_m}^2 + m\sum \frac{{m_i}^2}{m^2}{\sigma_m}^2 - 2\sum \frac{{m_i}^2}{m}{\sigma_m}^2 \\ &= {\sigma_m}^2\left(m - \sum \frac{{m_i}^2}{m}\right)\end{aligned}$$

72

In total the aggregate claims were 15.7 million, and the total exposure was 14,693 lives. The average claim per life is therefore $\frac{15700000}{14693} = 1068.54$. The averages for the three companies are:

$$\begin{aligned}\frac{5300000}{3623} &= 1,462.88\\ \frac{4000000}{4908} &= \quad 815.00\\ \frac{6400000}{6162} &= 1,038.62\end{aligned}$$

The variances for the three companies are:

$$\frac{769(1690.51-1462.88)^2+928(1616.38-1462.88)^2+880(909.09-1462.88)^2+1046(1625.24-1462.88)^2}{3} = 116443575$$
$$\frac{1430(699.30-815)^2+1207(745.65-815)^2+949(632.24-815)^2+1322(1134.64-815)^2}{3} = 63905244$$
$$\frac{942(1167.73-1038.62)^2+1485(942.76-1038.62)^2+2031(935.50-1038.62)^2+1704(1173.71-1038.62)^2}{3} = 27347095$$

The expected process variance is therefore:

$$\frac{3623\times 116443575+4908\times 63905244+6162\times 27347095}{14693} = 61528266$$

We have that

$$3623(1462.88-1068.54)^2+4908(815.00-1068.54)^2+6162(1038.62-1068.54)^2 = 884406185$$

We have that

$$884406185 - 2\times 61528266 = 761349653$$

We also get

$$m - \sum \frac{{m_i}^2}{m} = 14693 - \frac{3623^2+4908^2+6162^2}{14693} = 9575.949$$

The variance of hypothetical means is $\frac{761349653}{9575.949} = 79506.44$

The credibilities of the three companies' experiences are therefore

$$\begin{aligned}Z_1 &= \frac{3623}{3623+\frac{61528266}{79506.44}} = 0.8239938\\ Z_2 &= \frac{4908}{4908+\frac{61528266}{79506.44}} = 0.8637989\\ Z_3 &= \frac{6162}{6162+\frac{61528266}{79506.44}} = 0.8884240\end{aligned}$$

The credibility premiums per unit of exposure are therefore:

$$
\begin{aligned}
0.8239938 \times 1462.88 + 0.1760062 \times 1068.54 &= \$1,393.47 \\
0.8637989 \times 815.00 + 0.1362011 \times 1068.54 &= \quad \$849.53 \\
0.8884240 \times 1038.62 + 0.1115760 \times 1068.54 &= \$1,041.96
\end{aligned}
$$

The credibility-weighted average is

$$
\frac{0.8239938 \times 1462.88 + 0.8637989 \times 815.00 + 0.8884240 \times 1038.62}{0.8239938 + 0.8637989 + 0.8884240} = \$1,099.34
$$

Using this average, the credibility premiums are

$$
\begin{aligned}
0.8239938 \times 1462.88 + 0.1760062 \times 1099.34 &= \$1,398.89 \\
0.8637989 \times 815.00 + 0.1362011 \times 1099.34 &= \quad \$853.73 \\
0.8884240 \times 1038.62 + 0.1115760 \times 1099.34 &= \$1,045.39
\end{aligned}
$$

19.3 Semiparametric Estimation

73

There are a total of 3193 claims from 6210 policyholders, so the estimate for μ is $\frac{3193}{6210} = 0.5141707$. Since for a Poisson distribution the mean and variance are equal, this gives the expected process variance is also $v = 0.5141707$. We calculate the sample variance

$$\frac{6210}{6209}\left(\frac{1406 + 740 \times 4 + 97 \times 9 + 13 \times 16 + 3 \times 25}{6210} - 0.5141707^2\right) = 0.6249401$$

so the variance of hypothetical means is $0.6249401 - 0.5141707 = 0.1107694$ and the credibility of 3 years of experience is

$$Z = \frac{3}{3 + \frac{0.5141707}{0.1107694}} = 0.3925771$$

so the credibility estimate is

$$0.3925771 \times 2 + 0.6074229 \times 0.5141707 = 1.097473$$

74

42590 claims were made in 91221 years by 34285 policyholders.

The global mean is therefore $\mu = \frac{42590}{91221} = 0.4668881$ claims per year.

For the Poisson distribution, the mean is equal to the variance, so the expected process variance is also 0.4668881.

As in Question 72, the estimator for VHMs is

$$\widehat{\text{VHM}} = \frac{\sum m_i \left(\frac{n_i}{m_i} - 0.4668881\right)^2 - (n-1)\text{EPV}}{m - \frac{\sum m_i{}^2}{m}}$$

We compute

$$\begin{aligned}
\sum m_i \left(\frac{n_i}{m_i} - 0.4668881\right)^2 &= 3951(0-0.4668881)^2 + 1406(1-0.4668881)^2 + 740(2-0.4668881)^2 + 97(3-0.4668881)^2 \\
&\quad + 13(4-0.4668881)^2 + 3(5-0.4668881)^2 + 2\left(3628(0-0.4668881)^2 + 2807(0.5-0.4668881)^2\right. \\
&\quad + 1023(1-0.4668881)^2 + 461(1.5-0.4668881)^2 + 104(2-0.4668881)^2 + 13(2.5-0.4668881)^2 \\
&\quad \left.+4(3-0.4668881)^2 + (4-0.4668881)^2\right) + 3\left(2967(0-0.4668881)^2\right. \\
&\quad + 4032(0.33333333-0.4668881)^2 + 2214(0.66666667-0.4668881)^2 + 890(1-0.4668881)^2 \\
&\quad + 734(1.33333333-0.4668881)^2 + 215(1.666666667-0.4668881)^2 + 131(2-0.4668881)^2 \\
&\quad \left.+22(2.33333333-0.4668881)^2 + 2(3-0.4668881)^2\right) + 4\left(1460(0-0.4668881)^2\right. \\
&\quad + 2828(0.25-0.4668881)^2 + 2204(0.5-0.4668881)^2 + 985(0.75-0.4668881)^2 \\
&\quad + 747(1-0.4668881)^2 + 358(1.25-0.4668881)^2 + 194(1.5-0.4668881)^2 + 43(1.75-0.4668881)^2 \\
&\quad \left.+8(2-0.4668881)^2\right) \\
&= 19670.9022002
\end{aligned}$$

$$m - \frac{\sum m_i{}^2}{m} = 91221 - \frac{6210 \times 1^2 + 8041 \times 2^2 + 11207 \times 3^2 + 8827 \times 4^2}{91221} = 91217.92539$$

So $\widehat{\text{VHM}} = \frac{19670.9022002 - 34284 \times 0.4668881}{91217.92539} = 0.0401687559121$

Because the different policyholders have different exposure, we should use a credibility-weighted average here. We calculate the credibility for different numbers of years of history:

Years	Credibility	Total claims	Total policyholders
1	$\frac{1}{1+\frac{0.4668881}{0.0401687559121}} = 0.079219431596$	3244	6210
1	$\frac{2}{2+\frac{0.4668881}{0.0401687559121}} = 0.146808756915$	6749	8041
1	$\frac{3}{3+\frac{0.4668881}{0.0401687559121}} = 0.205153938061$	16099	11207
1	$\frac{4}{4+\frac{0.4668881}{0.0401687559121}} = 0.256030059119$	16498	8827

The credibility weighted average is therefore

$$\frac{0.079219431596 \times 3244 + 0.146808756915 \times \frac{6749}{2} + 0.205153938061 \times \frac{16099}{3} + 0.256030059119 \times \frac{16498}{4}}{0.079219431596 \times 6210 + 0.146808756915 \times 8041 + 0.205153938061 \times 11207 + 0.256030059119 \times 8827} = 0.466866294171$$

so the credibility estimate is

$$0.205153938061 \times 0.66666666667 + 0.794846061939 \times 0.466866294171 = 0.507856127415$$

The expected number of claims is therefore $0.507856127415 \times 64 = 32.5027921546$.

75

The total exposure is $45+10+45+14+27+12+74+27+10+293+14+13+10+14+17+6=631$ units.

The means for each individual are: $\frac{34}{114}=0.2982456$, $\frac{0}{140}=0$, $\frac{169}{330}=0.5121212$, and $\frac{7}{47}=0.1489362$. The average value of λ (Using equal weighting of policyholders) is therefore $\frac{0.2982456+0+0.5121212+0.1489362}{4}=0.2398258$, so this is the expected process variance, because the variance of a Poisson distribution is equal to the mean.

Suppose the hypothetical means are λ_1, λ_2, λ_3, and λ_4, and denote our estimated means by $\widehat{\lambda_1}$, $\widehat{\lambda_2}$, $\widehat{\lambda_3}$, and $\widehat{\lambda_4}$. We have that $\mathbb{E}(\widehat{\lambda_i}|\lambda_i)=\lambda_i$ and $\mathrm{Var}(\widehat{\lambda_i}|\lambda_i)=\frac{\lambda_i}{n_i}$. Letting $\widehat{\lambda}=\frac{\widehat{\lambda_1}+\widehat{\lambda_2}+\widehat{\lambda_3}+\widehat{\lambda_4}}{4}$, we have that $\frac{(\widehat{\lambda_1}-\widehat{\lambda})^2+(\widehat{\lambda_2}-\widehat{\lambda})^2+(\widehat{\lambda_3}-\widehat{\lambda})^2+(\widehat{\lambda_4}-\widehat{\lambda})^2}{3}$ is an unbiased estimator for $\mathrm{Var}(\widehat{\lambda_i})$. Now we have that

$$\mathrm{Var}(\widehat{\lambda_i})=\mathrm{Var}(\mathbb{E}(\widehat{\lambda_i}|\lambda_i))+\mathbb{E}(\mathrm{Var}(\widehat{\lambda_i}|\lambda_i))=\mathrm{Var}(\lambda_i)+\mathbb{E}\left(\frac{\lambda_i}{n_i}\right)$$

We calculate

$$\begin{aligned}\mathrm{Var}(\widehat{\lambda_i})&\frac{(0.2982456-0.2398258)^2+(0-0.2398258)^2+(0.5121212-0.2398258)^2+(0.1489362-0.2398258)^2}{3}\\&=0.04777833\end{aligned}$$

We also calculate

$$\mathbb{E}\left(\frac{\lambda_i}{n_i}\right)=\frac{1}{4}\left(\frac{0.2398258}{114}+\frac{0.2398258}{140}+\frac{0.2398258}{330}+\frac{0.2398258}{47}\right)=0.00241154949013$$

The VHM is therefore $0.04777833-0.00241154949013=0.0453667805099$.

The credibility for the four policyholders is therefore given in the following table:

Policyholder	Exposure	Credibility
1	114	$\frac{114}{114+\frac{0.2398258}{0.0453667805099}}=0.955683331831$
2	140	$\frac{140}{140+\frac{0.2398258}{0.0453667805099}}=0.963614105621$
3	330	$\frac{330}{330+\frac{0.2398258}{0.0453667805099}}=0.984233255261$
4	37	$\frac{37}{37+\frac{0.2398258}{0.0453667805099}}=0.874986334869$

To balance the estimates, we set the book premium to equal the credibility-weighted mean

$$\frac{0.955683331831\times0.2982456+0.963614105621\times0+0.984233255261\times0.5121212+0.874986334869\times0.1489362}{0.955683331831+0.963614105621+0.984233255261+0.874986334869}=0.24332091069$$

The credibility estimate for λ_3 is therefore $0.984233255261\times0.5121212+0.015766744739\times0.243320910699=0.507883094453$. This means the expected number of claims for policyholder 3 is $0.507883094453\times64=32.504518045$.

76 For an exponential distribution, the variance is the square of the mean, so the estimate for EPV is the average square of the hypothetical means. This is the square of the mean claim amount plus the variance of hypothetical means. We have that the variance of observed means is 832^2. This is the VHM plus the EPV. Since EPV $=$ VHM $+ 689^2$, we have that $2\text{VHM} + 689^2 = 832^2$, so VHM $= \frac{832^2-689^2}{2} = 108751.5$ and EPV $= 108751.5 + 689^2 = 583472.5$. The credibility of one year's experience is therefore $Z = \frac{1}{1+\frac{583472.5}{108751.5}} = 0.157104492188$. The premium for this individual is therefore $0.157104492188 \times 462 + 0.842895507812 \times 689 =$ \$653.34.

IRLRPCI 4 Loss Reserving

4.6 Loss Reserving Methods

77

We use the following code:

```
Run.Off<-read.table("RunOff1.txt")
Cum.Run.Off<-t(apply(Run.Off,1,cumsum))
### Take cumulative sums along rows.
### apply automatically returns its answers as columns, so we need to transpose

### Development factors for each development year and accident year
### Check these for outliers.
Cum.Run.Off[,-1]/Cum.Run.Off[,-6]

Cum.Cum.Payments<-apply(Cum.Run.Off,2,cumsum)

### In R matrices are vectors index by row then column,
### so we can extract the antidiagonal elements by index
Cum.Cum.Payments[1+seq_len(5)*5]

### Mean development factors are obtained by dividing adjacent
### elements in this table.
Dev.Factor<-Cum.Cum.Payments[6+seq_len(5)*5]/Cum.Cum.Payments[seq_len(5)*5]

### To get the ultimate Development factors we can use cumulative products.
### Reverse the list so it is indexed by starting DY.
Ultimate.Dev.Factor<-rev(cumprod(rev(Dev.Factor)))

### Latest Cumulative payments
Cum.Run.Off[1+seq_len(5)*5]

### Expected Ultimate Losses
rev(Cum.Run.Off[1+seq_len(5)*5]*Ultimate.Dev.Factor)

### Expected Outstanding Claims
rev(Cum.Run.Off[1+seq_len(5)*5]*(Ultimate.Dev.Factor-1))
sum(Cum.Run.Off[1+seq_len(5)*5]*(Ultimate.Dev.Factor-1))

### Can also break down by year:
### Use matrix multiplication of rows.
### This fills estimates for known values, which need to be ignored.
### Reverse the payments made vector to put the table in the correct order.
rev(Cum.Run.Off[1+seq_len(5)*5]*Ultimate.Dev.Factor)%*%t(c(1/Ultimate.Dev.Factor,1))

### Note the complete AY0 has been removed.
```

```
### To get expected future payments, take differences between consecutive years:

rev(Cum.Run.Off[1+seq_len(5)*5]*Ultimate.Dev.Factor)%*%
    t(c(1/Ultimate.Dev.Factor[-1],1)-1/Ultimate.Dev.Factor)
```

This gives the following table of (rounded) expected future payments:

Accident	Development Year				
Year	1	2	3	4	5
1					453
2				1531	473
3			2068	1609	497
4		2804	2178	1694	523
5	2631	2886	2242	1744	539

78
We use the following code to adjust the historical data to Year 5 costs:

```
### Calculate an inflation factor to year 5 from each previous year.

Inflation.Factor<-rev(cumprod(rev(c(1.02,1.04,1.07,1.05,1.01,1))))

### Calculate the year of each entry
Year<-(0:5)%*%t(rep(1,6))+rep(1,6)%*%t(0:5)

### Use these as indices in the Inflation Factor vector to get
### inflation factors for each element of the table.  Since year
### numbering starts at 0, but vector indices start at 1, we need to
### add 1 to get the correct factor.

Inflation.Factor[Year+1]

### This lookup operation creates a vector, so we need to turn it back
### into a matrix.
matrix(Inflation.Factor[Year+1],6,6)

### Adjust payments. Do this before calculating cumulative payments.

Adj.Run.Off<-Run.Off*matrix(Inflation.Factor[Year+1],6,6)
```

Now we can use the method from Question 77 to predict future losses, getting the following future payments (assuming no inflation).

Accident	Development Year				
Year	1	2	3	4	5
1					435
2				1441	434
3			1934	1446	436
4		2646	1958	1464	441
5	2535	2659	1968	1471	443

If we have expected future inflation rates, we can also adjust payments by these rates in a similar fashion.

79

We use the following code to calculate correlations:

```
### Calculate a matrix of annual development factors
Annual.Dev.Factors<-Cum.Run.Off[,-1]/Cum.Run.Off[,-6]

### calculate pairwise correlations for each pair of years.
### We exclude NAs from the correlation calculations
cor(Annual.Dev.Factors[,seq_len(3)],use="pairwise.complete.obs")
```

The only meaningful correlations are from pairs of development years that have at least 3 observations — i.e. Years 0–1, 1–2, and 2–3. We get the correlation between Years 0–1 and 1–2 is −0.57937360 and the correlation between Years 1–2 and 2–3 is 0.00691387. Under a normal assumption, we can perform a t-test. For comparing factors 0–1 and 1–2, the test statistic $T_1 = -0.57937360\sqrt{4-21-0.57937360^2} = -0.66782765681$ should follow a t distribution with $4-2=2$ degrees of freedom. The p-value is 0.5729913. For comparing factors 0–1 and 1–2, the test statistic $T_2 = 0.00691387\sqrt{3-21-0.00691387^2} = 0.00691370475101$ should follow a t distribution with $3-2=1$ degrees of freedom. The p-value is 0.9955987. Thus, we do not have any evidence for correlation of development factors.

The Spearman coefficients are −1 and 0.5. We get $T_2 = 0.5\sqrt{3-21-0.5^2} = 0.433012701892$, which is not significant. For T_1, the t statistic becomes infinite, so does not give a reliable p-value. We can get the correct p-value by considering permutations — there are two possible orders that achieve a Spearman correlation coefficient of ±1, out of a total of 12 possible orders, so the p-value is $\frac{1}{12} = 0.083333333333$.

We can test for calendar year effects by ranking within each column.

```
### We first rank each column.
DFRanks<-apply(Annual.Dev.Factors,2,rank)

### Note that this ranks NAs last, so the NA elements get false ranks.

### rescale ranks by dividing by number of elements
### plus 1 so that the median is 0.5
DFRanks.scaled<-DFRanks/(rep(1,6)%*%t(colSums(!is.na(Annual.Dev.Factors)))+1)


### We can arrange by Calendar years by changing the dimensions of the matrix:

Cal.Year.DF.Ranks<-matrix(DFRanks.scaled,5,6)

### Also note that the false ranks are all > 1 after rescaling.
### So we can replace them by NA

Cal.Year.DF.Ranks[Cal.Year.DF.Ranks>=1]<-NA

### Collect Binomial statistics for simple test:

cbind(rowSums(Cal.Year.DF.Ranks<0.5,na.rm=TRUE),
      rowSums(!is.na(Cal.Year.DF.Ranks)))
```

```
### A more powerful test would calculate the average of these rescaled
### ranks by calendar year. However, calculating significance is
### a slight challenge.
cbind(rowMeans(Cal.Year.DF.Ranks,na.rm=TRUE),
      rowSums(!is.na(Cal.Year.DF.Ranks)))
```

We get the following:

Calendar Year	No. of Development Factors	Average Rank	Number of factors below median
1	1	0.1666667	1
2	2	0.5666667	1
3	3	0.6444444	0
4	4	0.4458333	2
5	5	0.4966667	2

For the average ranks, the rank in development year i, which has $I-i$ observations has mean 0.5, but the variance depends on the number of observations, $I-i$. With $n+1$ observations, the variance of the scaled ranks is $\frac{1}{12}\frac{n}{n+2}$. We can therefore rescale the ranks by the standard deviations to get unit variance, and the average rank should have variance equal to the number of development factors.

This gives the following

Calendar Year	No. of Development Factors	Sum of Standardised Ranks
1	1	-1.41421356
2	2	0.63453401
3	3	1.86142716
4	4	-1.11689592
5	5	0.03514831

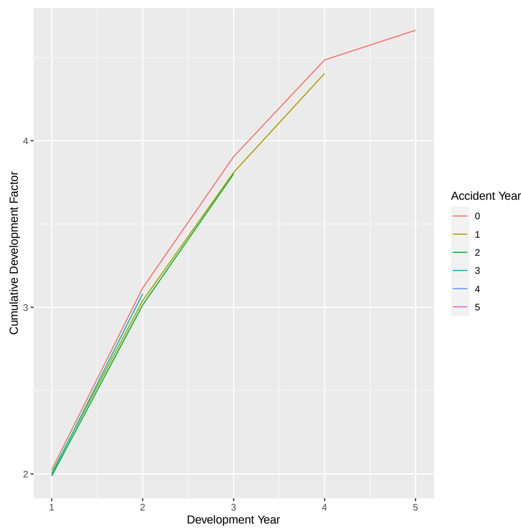
We see that none of these tests can reject the null hypothesis, even without correction for multiple testing.

80 We use the following code:

```
library(dplyr)  # for %>%
library(reshape) # for melt - turns matrix into table
library(ggplot2) # for the plotting

ggplot((Cum.Adj.Run.Off[,-1]/(Cum.Adj.Run.Off[,1]%*%t(rep(1,5))))%>%melt(),
       mapping=aes(x=as.numeric(X2),  # development year
                   y=value,  # CDF
                   colour=as.factor(X1-1)))+ # Accident year
    geom_line()+
    scale_x_continuous(name="Development Year")+
    scale_y_continuous(name="Cumulative Development Factor")+
    scale_colour_discrete(name="Accident Year")
```

This produces the following diagnostic plot.



We see that Accident Year 0 has slightly higher adjusted cumulative development factors, but there is not a general trend, and the difference is small enough to be random fluctuation.

81

First we calculate the expected Loss payments. Using the loss development factors, the proportion of payments made in each year is:

Cumulative	0.1995275	0.4092979	0.6393469	0.8180704	0.9570632	1
Proportion	0.19952752	0.20977035	0.23004908	0.17872344	0.13899278	0.04293684

This leads to expected payments:

Accident	Expected	Development Year					
Year	loss	0	1	2	3	4	5
1	9805						421
2	10214					1420	439
3	10724				1917	1491	460
4	11640			2678	2080	1618	500
5	11826		2481	2721	2114	1644	508

We can calculate these using the code:

```
### Calculate gamma_j

gamma<-c(1/rev(cumprod(rev(Dev.Factor))),1)-c(0,1/rev(cumprod(rev(Dev.Factor))))

### Earned premiums
EP<-c(11980,12105,12610,13240,14370,14600)

### Earned premium * Expected Loss Ratio * Proportion of claims paid
(EP*0.81)%*%t(gamma)
```

82 We first convert the run-off table to the following cumulative per-premium run-off table.

Accident	Development Year					
Year	0	1	2	3	4	5
0	0.1674457	0.3425710	0.5363105	0.6861436	0.8019199	0.8378965
1	0.1738125	0.3558860	0.5556382	0.7120198	0.8337877	
2	0.1741475	0.3582078	0.5590801	0.7144330		
3	0.1747734	0.3580060	0.5588369			
4	0.1687543	0.3471120				
5	0.1714384					

Using the same method as Question 77, we get the following estimated cumulative LDFs and ultimate losses.

Accident Year i	$\hat{C}_{i,J}$	β_{J-i}	γ_{J-j}
0	0.8378965	1.0000000	0.04293684
1	0.8711940	0.9570632	0.13899278
2	0.8733148	0.8180704	0.17872344
3	0.8740745	0.6393469	0.23004908
4	0.8480671	0.4092979	0.20977035
5	0.8592216	0.1995275	0.19952752

This gives $\hat{v} = 1.111976 \times 10^{-6}$, $\overline{C} = 0.8608603$ and $\hat{a} = 0.0002881496$. These result in the following credibility estimates:

Accident Year	Credibility
0	0.9961558
1	0.9959840
2	0.9953049
3	0.9940003
4	0.9906597
5	0.9810262

The credibility weighted average ultimate losses are therefore 0.8606363, and the Bühlmann-Straub estimate for Ultimate losses in each year is

i	$\hat{C}_{i,J}^{\text{BS}}$	Estimated per-premium Outstanding Claims	Estimated outstanding claims
0	0.8379839	0.0000000	0.0000
1	0.8711516	0.0374045	452.7814
2	0.8732553	0.1588710	2003.3633
3	0.8739939	0.3152086	4173.3613
4	0.8481845	0.5010244	7199.7202
5	0.8592485	0.6878047	10041.9493

The total estimated outstanding claims are therefore 23871.18.

SN1 3.5 The Poisson Model

83 For given μ_i and γ_j, the log-likelihood is

$$l(\mu\gamma) = \sum_{i+j\leqslant I} X_{i,j}\log(\mu_i\gamma_j) - \mu_i\gamma_j$$

Setting the derivative with respect to μ_i to zero gives

$$\sum_{j\leqslant I-i} \frac{X_{i,j}}{\mu_i} - \gamma_j = 0$$

so

$$\hat{\mu}_i = \frac{\sum_{j=0}^{I-i} X_{i,j}}{\sum_{j=0}^{I-i}\gamma_j} = \frac{C_{i,I-i}}{\beta_{I-i}} \tag{1}$$

Setting the derivative with respect to γ_j to zero gives

$$\sum_{i\leqslant I-j} \frac{X_{i,j}}{\gamma_j} - \mu_i = 0$$

so

$$\gamma_j = \frac{\sum_{i=0}^{I-j} X_{i,j}}{\sum_{i=0}^{I-j}\mu_i} \tag{2}$$

The MLE satisfies these equations and $\sum_{j=0}^{J}\gamma_j = 1$. We will now prove that $\beta_{J-j} = \frac{\sum_{i=0}^{I-J+j} C_{i,J-j}}{\sum_{i=0}^{I-J+j}\mu_i}$ by induction on j. We will need the following lemma:

Lemma 2. *If $\frac{A}{B} = \frac{C}{D}$ and $A \neq C$, then $\frac{A+C}{B+D} = \frac{A}{B}$.*

Proof.

$$\frac{A+C}{B+D} = \frac{A+\frac{AD}{B}}{B+D} = \frac{A}{B+D}\left(1+\frac{D}{B}\right) = \frac{A}{B+D}\frac{B+D}{B} = \frac{A}{B}$$

□

For the base step, when $j=0$, this is immediate from the fact that $\sum_{j=0}^{J}\gamma_j = 1$. Suppose we have proved $\beta_{J-j} = \frac{\sum_{i=0}^{I-J+j} C_{i,J-j}}{\sum_{i=0}^{I-J+j}\mu_i}$. We want to prove the same results for $j+1$.

We have

$$\beta_{J-j-1} = \beta_{J-j} - \gamma_{J-j} = \frac{\sum_{i=0}^{I-J+j} C_{i,J-j}}{\sum_{i=0}^{I-J+j}\mu_i} - \frac{\sum_{i=0}^{I-J+j} X_{i,J-j}}{\sum_{i=0}^{I-J+j}\mu_i} = \frac{\sum_{i=0}^{I-J+j} C_{i,J-j-1}}{\sum_{i=0}^{I-J+j}\mu_i}$$

Furthermore, from (1), we have $\beta_{J-j-1} = \frac{C_{I-J+j+1,J-j-1}}{\mu_{I-J+j+1}}$, so by the lemma,

$$\beta_{J-j-1} = \frac{C_{I-J+j+1,J-j-1} + \sum_{i=0}^{I-J+j} C_{i,J-j-1}}{\mu_{I-J+j+1} + \sum_{i=0}^{I-J+j}\mu_i} = \frac{\sum_{i=0}^{I-J+j+1} C_{i,J-j-1}}{\sum_{i=0}^{I-J+j+1}\mu_i}$$

as required.

From this, we have

$$\frac{\hat{\gamma}_j}{\hat{\beta}_j} = \frac{\left(\frac{\sum_{i=0}^{I-j} X_{i,j}}{\sum_{i=0}^{I-j} \mu_i}\right)}{\left(\frac{\sum_{i=0}^{I-j} C_{i,j}}{\sum_{i=0}^{I-j} \mu_i}\right)} = \frac{\sum_{i=0}^{I-j} X_{i,j}}{\sum_{i=0}^{I-j} C_{i,j}}$$

so

$$\frac{\hat{\beta}_{j-1}}{\hat{\beta}_j} = 1 - \frac{\sum_{i=0}^{I-j} X_{i,j}}{\sum_{i=0}^{I-j} C_{i,j}} = \frac{\sum_{i=0}^{I-j} C_{i,j} - \sum_{i=0}^{I-j} X_{i,j}}{\sum_{i=0}^{I-j} C_{i,j}} = \frac{\sum_{i=0}^{I-j} C_{i,j-1}}{\sum_{i=0}^{I-j} C_{i,j}}$$

which is exactly the chain-ladder estimate.

The chain-ladder estimate is $\hat{\beta}_{j-1} = \frac{\sum_{i=0}^{I-j} C_{i,j-1}}{\sum_{i=0}^{I-j} C_{ij}} \hat{\beta}_j = \left(1 - \frac{\sum_{i=0}^{I-j} X_{ij}}{\sum_{i=0}^{I-j} C_{i,j}}\right) \hat{\beta}_j$, and $\hat{\beta}_J = 1$. Under the Poisson model, $\sum_{i=0}^{I-j} X_{ij}$ is Poisson with mean $\gamma_j \sum_{i=1}^{I-j} \mu_i$, so $\mathrm{Var}(\hat{\beta}_j | \hat{\beta}_{j-1}, C_{i,j-1}) = \frac{\hat{\beta}_{j-1}^2}{\left(\sum_{i=0}^{I-j} C_{i,j-1}\right)^2} \gamma_j \sum_{i=1}^{I-j} \mu_i$.

(a) We use the following code to calculate $\hat{\sigma_j}^2$:

```
### Annual development factors
fij <-Cum.Run.Off[,-1]/Cum.Run.Off[,-6]

### Use the rowMeans function to automatically adjust for NA values,
### then correct the scale factors.
sigmahat<-rowMeans(Cum.Run.Off[,-6]*(fij-rep(1,6)%*%t(Dev.Factor))^2,na.rm=TRUE)*
    (6-seq_len(6))/(5-seq_len(6))

### Use Mack's suggestion for estimating sigma_{J-1}
sigmahat[5]<-min(c(sigmahat[3:4],sigmahat[4]^2/sigmahat[3]))
```

This gives the following estimates:

j	$\hat{\sigma}_j^2$
0	0.03035242
1	0.02272968
2	0.04781808
3	0.02552981
4	0.01363023

(b) The process variance can be approximated by

$$\mathrm{Var}(C_{i,J}|C_{i,I-i}) \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} \hat{C}_{i,j}} = \hat{C}_{i,J} \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} \beta_j}$$

We calculate these using the following code:

```
### calculate the sums
cumsum(rev(sigmahat[-6]*Ultimate.Dev.Factor/Dev.Factor^2)))

### Multiply by estimated ultimate losses
### Remove the first row as those are final.
cumsum(rev(sigmahat[-6]*Ultimate.Dev.Factor/Dev.Factor^2)))*Est.Ult.Losses[-1]
```

This gives the following values:

Accident Year i	$\sum_{j=J-i}^{J} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \hat{\beta}_j}$	$\hat{C}_{i,J}$	$\hat{\mathrm{Var}}(C_{i,J}|C_{i,I-i})$
1	0.01304499	0.8711940	0.01136472
2	0.03584616	0.8733148	0.03130498
3	0.08152840	0.8740745	0.07126189
4	0.10428773	0.8480671	0.08844299
5	0.14043847	0.8592216	0.12066777
Total			0.3230423

(c) The expected squared estimation errors are given by

$$\mathbb{E}\left(\left(\hat{C}_{i,J} - \mathbb{E}(C_{i,j}|D_I)\right)^2\right) \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} S_j}$$

We therefore use the following code to compute the mean squared estimation errors:

```
### Easiest to compute S by taking total sums then subtracting anti-diagonal.
S<-colSums(Cum.Run.Off,na.rm=TRUE)-Cum.Run.Off[1+5*seq_len(6)]

### MSE of hat{C}_{i,J}
Est.Ult.Losses[-1]^2*cumsum(rev(sigmahat[-6]/(Dev.Factor^2*S[-6])))
```

This gives the following values:

Accident Year i	$\mathbb{E}\left(\left(\hat{C}_{i,J} - \mathbb{E}(C_{i,j}\|D_I)\right)^2\right)$
1	9.863391×10^{-7}
2	1.835985×10^{-6}
3	2.943788×10^{-6}
4	3.150395×10^{-6}
5	3.715945×10^{-6}

(d)

$$\mathbb{E}\left(\left(\hat{C}_{i,J} - \mathbb{E}(C_{i,j}|D_I)\right)\left(\hat{C}_{i',J} - \mathbb{E}(C_{i',j}|D_I)\right)\right) \approx \hat{C}_{i,J}\hat{C}_{i',J} \sum_{j=I-(i\wedge i')}^{J} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2} S_j}$$

where $S_j = \sum_{i=0}^{I-1-j} C_{i,j}$.

We therefore use the following code to compute the mean squared estimation errors:

```
### Covariance of estimation errors
CovarianceMSE<-(Est.Ult.Losses[-1]%*%t(Est.Ult.Losses[-1]))*
    pmax(
        rev(cumsum(rev(sigmahat[-6]/(Dev.Factor^2*S[-6]))))%*%t(rep(1,5)),
        rep(1,5)%*%t(rev(cumsum(rev(sigmahat[-6]/(Dev.Factor^2*S[-6]))))))

### Total estimation error.
sum(CovarianceMSE)+sum(Est.Ult.Losses[-1]^2*cumsum(rev(sigmahat[-6]/(Dev.Factor^2*S[-6]))))
```

This gives the following values:

i	j				
	1	2	3	4	5
1	3.820222×10^{-6}	3.829522×10^{-6}	3.832853×10^{-6}	3.718809×10^{-6}	3.767722×10^{-6}
2	3.829522×10^{-6}	3.340768×10^{-6}	3.343674×10^{-6}	3.244185×10^{-6}	3.286856×10^{-6}
3	3.832853×10^{-6}	3.343674×10^{-6}	2.943788×10^{-6}	2.856197×10^{-6}	2.893765×10^{-6}
4	3.718809×10^{-6}	3.244185×10^{-6}	2.856197×10^{-6}	1.731361×10^{-6}	1.754134×10^{-6}
5	3.767722×10^{-6}	3.286856×10^{-6}	2.893765×10^{-6}	1.754134×10^{-6}	9.594158×10^{-7}

The mean squared estimation error in the total outstanding claims is therefore 7.748614×10^{-5}.

The total MSE is obtained by adding this to the process variance, which gives $7.748614 \times 10^{-5} + 0.3230423 = 0.32311978614$.

85 We use the following code

```
library(reshape)
Run.Off.Matrix<-as.matrix(Run.Off)
rownames(Run.Off.Matrix)<-paste("V",seq_len(6),sep="")
rownames(Run.Off.Matrix)<-paste("R",seq_len(6),sep="")
ODP<-glm(value~.,data=melt(Run.Off.Matrix),family=quasipoisson(link=log))
```

The fitted parameters are

Parameter	Estimate	Standard Error
(Intercept)	7.602330	0.002848
μ_1	0.049350	0.002943
μ_2	0.092653	0.003088
μ_3	0.142275	0.003321
μ_4	0.193969	0.003832
μ_5	0.222915	0.005014
γ_1	0.050061	0.002744
γ_2	0.142341	0.002947
γ_3	-0.110113	0.003536
γ_4	-0.361530	0.004554
γ_5	-1.536222	0.010345

These are log coefficients, and are relative to μ_0 and γ_0 respectively, so we need to rescale to ensure $\sum_{j=0}^{J} \gamma_j = 1$.

(a) We use the following code:

```
Run.Off.Reported<-read.table("ClaimsReportedRunOff.txt")
Run.Off.Settled<-read.table("ClaimsSettledRunOff.txt")
Aggregate.Payments<-read.table("AggregateSettledPaymentsRunOff.txt")

Cum.Reported<-t(apply(Run.Off.Reported,1,cumsum))
Cum.Cum.Reported<-apply(Cum.Reported,2,cumsum)
Reported.DF<-(Cum.Cum.Reported[,-1]/Cum.Cum.Reported[,-6])[5*seq_len(5)]
Est.Ultimate.Reported<-c(1,cumprod(rev(Reported.DF)))*Cum.Reported[rev(1+5*seq_len(6))]
Est.Cum.Reported<-Est.Ultimate.Reported%*%t(rev(1/c(1,cumprod(rev(Reported.DF)))))

Cum.Settled<-t(apply(Run.Off.Settled,1,cumsum))
Cum.Cum.Settled<-apply(Cum.Settled,2,cumsum)
Settled.DF<-(Cum.Cum.Settled[,-1]/Cum.Cum.Settled[,-6])[5*seq_len(5)]
beta<-c(rev(cumprod(rev(1/Settled.DF))),1)
gamma<-beta-c(0,beta[-6])

Projected.Settled<-Est.Ultimate.Reported%*%t(gamma)

Projected.Settled.Future<-Projected.Settled
Projected.Settled.Future[1,1:6]<-NA
Projected.Settled.Future[2,1:5]<-NA
Projected.Settled.Future[3,1:4]<-NA
Projected.Settled.Future[4,1:3]<-NA
Projected.Settled.Future[5,1:2]<-NA
Projected.Settled.Future[6,1]<-NA


Projected.Past.Settled<-matrix(rowMeans(cbind(
    as.vector(as.matrix(Run.Off.Settled)),
    as.vector(as.matrix(Projected.Settled.Future))),na.rm=TRUE),6,6)
```

This produces the following projection for settlements.

Accident	Development Year				
Year	1	2	3	4	5
1					4.025000
2				22.22409	3.875000
3			33.81494	26.48486	4.617910
4		33.75849	33.16157	25.97312	4.528682
5	59.69948	37.75756	37.08992	29.04992	5.065155

(b) We use the code:

```
Ave.Settlement.Ammount<-colMeans(Aggregate.Payments/Run.Off.Settled,na.rm=TRUE)
Exp.Agg.Payments<-Projected.Settled.Future*(rep(1,6)%*%t(Ave.Settlement.Ammount))
```

```
sum(Exp.Agg.Payments,na.rm=TRUE)
```

To project the following aggregate claims:

Accident	Development Year				
Year	1	2	3	4	5
1					22502.77
2				99324.95	21664.16
3			124618.7	118367.41	25817.58
4		75307.73	122210.8	116080.29	25318.73
5	70205.17	84228.77	136688.0	129831.30	28318.02

IRLRPCI 3 Ratemaking

3.9 Rate Changes

87 For the base class, the loss ratio is $\frac{3900}{4100} = 0.9512195$. We want to change the current differentials to match this loss ratio. For example, for the low risk class, at a differential of 0.74, we get a loss ratio of $\frac{1100}{1300} = 0.8461538$. That is, if the premium were the same as for the base class, the loss ratio would be $\frac{1100}{\left(\frac{1300}{0.74}\right)} 0.8461538 \times 0.74 = 0.6261538$. To get this loss ratio to equal 0.9512195, we would need the new differential to be the solution to

$$\frac{\left(0.74 \times \frac{1100}{1300}\right)}{d} = \frac{3900}{4100}$$

or

$$d = 0.74 \times \frac{1100}{1300} \times \frac{4100}{3900} = 0.6582643$$

Similarly for the high risk class, the new differential is

$$1.46 \times \frac{1400}{1600} \times \frac{4100}{3900} = 1.343013$$

88 Loss Ratio Method:

The new differentials are Low risk $0.74\times\frac{1900}{2000}\times\frac{9100}{8000}=0.7996625$ High risk $1.46\times\frac{2300}{3300}\times 91008000=1.157492$ Female : $0.88\times\frac{5850}{6900}\times\frac{7500}{6350}=0.8812051$.

The permissible loss ratio is $1-0.2=80\%$. At the current premium, the loss ratio is $\frac{12200}{14400}=0.8472222$, so if they used the same relative changes to premiums, the premiums would change by a factor of $\frac{0.8472222}{0.8}=1.059028$. However, we want to balance back by dividing by the off-balance factor which is

$$\frac{\frac{0.7997}{0.74}\times 900+1\times 4700+\frac{1.1575}{1.46}\times 1900+\frac{0.7997}{0.74}\times\frac{0.8812}{0.88}\times 1100+\frac{0.8812}{0.88}\times 4400+\frac{1.1575}{1.46}\times\frac{0.8812}{0.88}\times 1400}{900+4700+1900+1100+4400+1400}=0.9643522$$

We therefore multiply the base rate by $\frac{1.059028}{0.9643522}=1.123013$. The new base rate is $1.098176\times 46.30=\$50.85$. The rates for other classes are therefore shown in the following table

	Male	Female
Low	40.66	35.83
Medium	50.85	44.81
High	58.85	51.86

We now compare the calculated differentials with the experience:

Calculated Differential				Experience	
		Male	Female	Male	Female
Differential		1	0.8812051		
Low	0.7996625	0.7996625	0.7046667	0.9896748	0.5768390
Medium	1	1.0000000	0.8812051	1.0000000	0.8941463
High	1.157492	1.1574920	1.0199879	1.0570475	1.1572153

Loss cost method:

To calculate the new differentials, we calculate the loss cost per unit of exposure for each class. For example for female policyholders, the total loss was \$5,850, and there were \$1,100 of earned premiums at a rate of $46.30\times 0.74\times 0.88$, which corresponds to $\frac{1100}{46.30\times 0.74\times 0.88}=36.48357$ units of exposure. Similarly, the number of units of exposure for the other classes were:

	Male	Female	total
Low	26.2682	36.4836	62.7517
Medium	101.5119	107.9914	209.5032
High	28.1073	23.5349	51.6422
total	155.8874	168.0098	

This gives the loss costs as

Class	Loss cost
Low	30.27804
Medium	38.18557
High	44.53722
Male	40.73453
Female	34.81940

This gives the following:

Calculated Differential				Experience	
		Male	Female	Male	Female
Differential		1	0.8547883		
Low	0.7929184	0.7929184	0.6777774	0.9896748	0.5768390
Medium	1	1.0000000	0.8547883	1.0000000	0.8941463
High	1.166336	1.1663360	0.9969704	1.0570475	1.1572153

For these differentials and the exposures calculated above, the total earned premiums would be 295.624 times the base rate. The expected total losses are \$12,200, so the new base rate is $\frac{12200}{0.8\times 295.624} = \$51.59$, and the new premiums are

	Male	Female
Low	40.90	34.96
Medium	51.59	44.09
High	60.17	51.43

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If we balance back to the new differentials, the adjusted earned premiums are:

Female

	Healthy	Unhealthy	Total
Young	3600	$1800 \times \frac{1.57}{1.49} = 1896.64$	5496.64
Old	$7300 \times \frac{1.63}{1.74} = 6838.51$	$6900 \times \frac{1.63}{1.74} \times \frac{1.57}{1.49} = 6810.84$	13649.35
Total	10438.51	8707.49	19145.99

Male

	Healthy	Unhealthy	Total
Young	$3200 \times \frac{1.14}{1.18} = 3091.53$	$1700 \times \frac{1.14}{1.18} \times \frac{1.57}{1.49} = 1730.55$	4822.08
Old	$5300 \times \frac{1.14}{1.18} \times \frac{1.63}{1.74} = 4796.64$	$5800 \times \frac{1.14}{1.18} \times \frac{1.63}{1.74} \times \frac{1.57}{1.49} = 5530.99$	10327.63
Total	7888.17	7261.54	15149.71

Thus the adjusted total earned premiums are \$34295.70, which results in a loss ratio of 0.845587118544, so the base premium should be adjusted by a factor $\frac{0.845587118544}{0.8} = 1.05698389818$.