ACSC/STAT 4703, Actuarial Models II

FALL 2023
Toby Kenney

Homework Sheet 2

Model Solutions

Basic Questions

1. *An insurer models losses as following a distribution with distribution function $F(x) = 1-(1+x^4)^{-1}$. They find that $c_n = n^{\frac{1}{4}}$ and $d_n = n^{\frac{1}{4}}$ make the distribution of block maxima converge. What is the limiting distribution?*

   We have $P(M_n < c_nx + d_n) = F(c_nx + d_n)^n = \left(1 - \frac{1}{(c_nx+d_n)^4+1}\right)^n$. Substituting the given values gives

$$\begin{aligned}\log\left(P(M_n < c_nx + d_n)\right) &= n\log\left(1 - \frac{1}{n + c_nxn^{\frac{3}{4}} + c_n^2x^2n^{\frac{1}{2}} + c_n^3x^3n^{\frac{1}{4}} + c_n^4x^4 + 1}\right)\\ &= -\frac{n}{n + 4nx + 6nx^2 + 4nx^3 + nx^4 + 1} + O(n^{-1})\\ &= -\frac{1}{(1+x)^4 + n^{-1}} + O(n^{-1})\\ &= -(1+x)^{-4} + O(n^{-1})\end{aligned}$$

   Thus, the limiting distribution is Fréchet, with $\xi = 4$.

2. *An insurer models losses as following a distribution with survival function $S(x) = (7x+\cos(2\pi x))^{-1}$. What values of c_n and d_n make the distribution of block maxima converge, and what is the limiting distribution?*

   We have $nS(c_nx + d_n) = n(7c_nx + 7d_n + \cos(2c_n\pi x + 2d_n\pi))^{-1}$. We want this to converge for every x. For $x = 0$, we want $n(7d_n + \cos(2d_n\pi))^{-1}$ to converge to 1. Since $\cos(2\pi d_n)$ is much smaller than d_n, we see that $d_n = \frac{n}{7}$ satisfies this condition. Similarly, we see that for $c_n = an$ for a constant a,

$$\begin{aligned}
\frac{n}{7c_nx+7d_n+\cos(2c_n\pi x+2d_n\pi)} &= \frac{n}{7anx+n+\cos(2a\pi x+2n\pi)}\\
&= \frac{1}{7ax+1} - n\left(\frac{1}{7anx+n} - \frac{1}{7anx+n+\cos(2a\pi x)}\right)\\
&= \frac{1}{7ax+1} - n\left(\frac{\cos(2a\pi x)}{(7ax+1)n((7ax+1)n+\cos(2a\pi x))}\right)\\
&\to \frac{1}{7ax+1}
\end{aligned}$$

In particular, if $a=\frac{1}{7}$, then we have $nS(c_nx+d_n)\to(1+x)^{-1}$, so the limiting distribution of M_n is a Fréchet distribution with $\xi=1$ and the values $c_n=\frac{n}{7}$ and $d_n=\frac{n}{7}$ make the sequence converge.

3. *A loss follows a distribution from the MDA of a Fréchet distribution with* $\xi=0.4$. *A reinsurer estimates that the probability of the loss exceeding $500,000 is 0.006 and the probability of a loss exceeding $1,000,000 is 0.002. What is the expected payment on an excess-of-loss reinsurance contract of $1,000,000 over $1,000,000 for this loss.*

   Since the distribution of X is in the MDA of a Fréchet distribution with $\xi=0.4$, the excess-loss function converges to a generalised Pareto distribution with $\xi=0.4$. We also have $P(X-500000>500000|X>500000)=\frac{0.002}{0.006}=\frac{1}{3}$, which gives the scale parameter of the excess-loss distribution. We have that $X-500000|X>500000$ has a Pareto distribution with parameters $\alpha=\frac{1}{0.4}=2.5$ and θ given by solving

$$\begin{aligned}
\left(\frac{\theta}{\theta+500000}\right)^{2.5} &= \frac{1}{3}\\
\frac{\theta}{\theta+500000} &= 3^{-0.4}\\
\frac{\theta+500000}{\theta} &= 3^{0.4}\\
\frac{500000}{\theta} &= 3^{0.4}-1\\
\theta &= \frac{500000}{3^{0.4}-1} = 906050.575795
\end{aligned}$$

   For $x>500000$ we therefore have $S(x)=0.006\left(\frac{\theta}{\theta+x-500000}\right)^{2.5}$ The expected payment on the reinsurance is therefore

$$\begin{aligned}\int_{1000000}^{2000000} 0.006\left(\frac{\theta}{\theta+x-500000}\right)^{2.5}\,dx &= \int_{\theta+500000}^{\theta+1500000} 0.006\theta^{2.5}u^{-2.5}\,du\\ &= 0.006\theta^{2.5}\left[-\frac{u^{-1.5}}{1.5}\right]_{\theta+500000}^{\theta+1500000}\\ &= 0.004\theta^{2.5}\left((\theta+500000)^{-1.5}-(\theta+1500000)^{-1.5}\right)\\ &= \$1,037.24\end{aligned}$$

Standard Questions

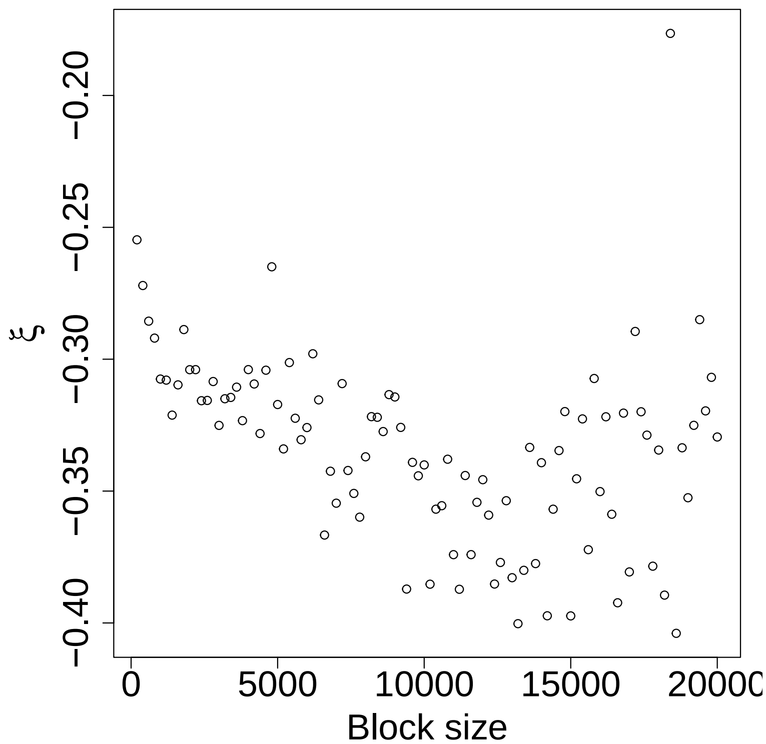
4. *The file* `HW2_data.txt` *contains 1,000,000 values of a random variable.*

   (a) *By dividing into blocks of different sizes, and using the* `fit.GEV` *function in the* `QRM` *package in* `R`*, estimate the tail index* ξ*.*

   We use the following `R` code to evaluate for block sizes multiples of 200 up to 20,000 (If the block size is too large, there are too few observations and `fit.GEV` produces an error):

```
HW2Q4<-read.table("../HW2_data.txt")[[1]]
library("QRM")
GEV_estimates<-rep(0,100)
for(i in seq_len(100)){
    nbl<-floor(5000/i)
    M<-matrix(HW2Q4[seq_len(nbl*200*i)],200*i,nbl)
    GEV_model<-fit.GEV(apply(M,2,max))
    GEV_estimates[i]<-GEV_model$par.ests["xi"]
}
```

   This produces the following estimates:

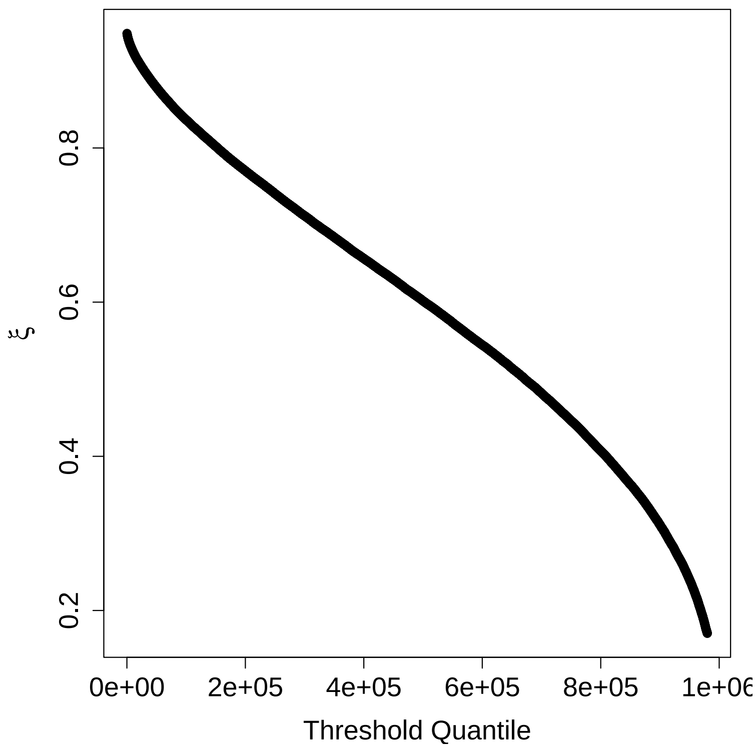


(b) *The file* `HW2_data.txt` *contains 1,000,000 values of a random variable. Use the Hill estimator to estimate* ξ *at a range of different thresholds.*

We use the following `R` code to evaluate for threshold positions multiples of 200 up to 980,000:

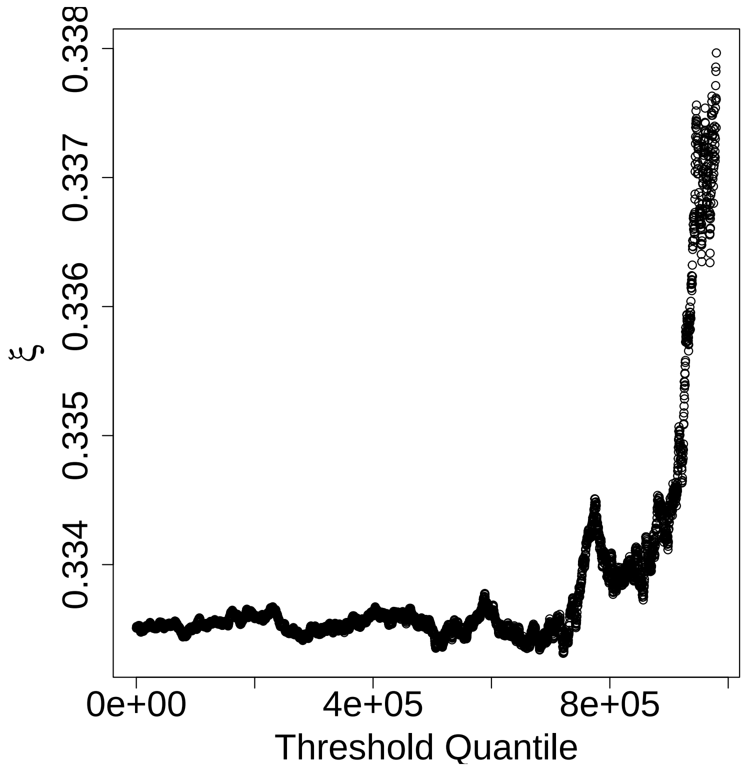
```
HW2Q4.log.sort<-sort(log(HW2Q4))
Hill_estimates<-rep(0,4900)
for(i in seq_len(4900)){
    pos<-i*200
    Hill_estimates[i]<-mean(HW2Q4.log.sort[(pos+1):1000000])-HW2Q4.log.sort[pos]
}
```

This produces the following estimates:



The reason the estimates do not converge well is that ξ is negative for the data, whereas the Hill estimator only works for positive ξ.

[I originally intended to include a second data set for this question with positive ξ. Using this dataset, the estimates produce the following plot, which shows a much more stable estimate, until the threshold gets too high, when the sample size becomes too small, leading to an unstable estimate.]



5. *A insurer wants to calculate the ILF for a heavy-tailed loss. Based on previous data, they estimate that the distribution of the loss is in the MDA of a Fréchet distribution with $\xi = 2$. The ILF from \$500,000 to \$1,000,000 is 1.28 and the ILF from \$500,000 to \$2,000,000 is 1.76. Assuming the GPD approximation applies to losses above \$500,000, what is the ILF from \$500,000 to \$5,000,000?*

   Under the GPD approximation, losses exceeding \$500,000 follow a GPD distribution with parameter ξ. The survival function is therefore $\left(1+\xi\frac{x}{\beta}\right)^{-\frac{1}{\xi}}$. We cannot use this approximation to estimate $\mathbb{E}(X \wedge 500000)$, but we have that

$$\begin{aligned}
\mathbb{E}(((X \wedge b) - a)|X > a) &= \int_0^{b-a} S_{x-a}(x)\,dx \\
&= \int_0^{b-a} \left(1 + 2\frac{x}{\beta}\right)^{-\frac{1}{2}}\,dx \\
&= \int_1^{1+\frac{2(b-a)}{\beta}} u^{-\frac{1}{2}}\,du \\
&= \left[\frac{2}{2-1}u^{1-\frac{1}{2}}\right]_1^{1+\frac{2(b-a)}{\beta}} \\
&= \frac{2}{2-1}\left(\left(1 + \frac{2(b-a)}{\beta}\right)^{\frac{2-1}{2}} - 1\right)
\end{aligned}$$

Let l_0 be the expected loss with policy limit \$500,000 and s_0 be the probability of a loss exceeding \$500,000. Since β is a scale parameter, we can rescale the loss in units of \$500,000. We have

$$\begin{aligned}
\mathbb{E}(((X\wedge 2)-1)_+) &= s_0\mathbb{E}(((X\wedge 2)-1)|X>1)\\
&= s_0\frac{2}{2-1}\left(\left(1+\frac{2}{\beta}\right)^{\frac{2-1}{2}}-1\right)=0.28l_0\\
\mathbb{E}(((X\wedge 4)-1)_+) &= s_0\frac{2}{2-1}\left(\left(1+\frac{6}{\beta}\right)^{\frac{2-1}{2}}-1\right)=0.76l_0\\
\left(\left(1+\frac{6}{\beta}\right)^{\frac{2-1}{2}}-1\right) &= \frac{0.76}{0.28}\left(\left(1+\frac{2}{\beta}\right)^{\frac{2-1}{2}}-1\right)\\
\sqrt{1+\frac{6}{\beta}} &= \frac{19}{7}\sqrt{1+\frac{2}{\beta}}-\frac{12}{7}\\
1+\frac{6}{\beta} &= \frac{361}{49}\left(1+\frac{2}{\beta}\right)-\frac{2\times 12\times 19}{49}\sqrt{1+\frac{2}{\beta}}+\frac{144}{49}\\
\frac{2\times 12\times 19}{49}\sqrt{1+\frac{2}{\beta}} &= \frac{361}{49}+\frac{144}{49}-1+\left(\frac{361}{49}-3\right)\frac{2}{\beta}=\frac{456}{49}+\frac{214}{49}\frac{2}{\beta}\\
\sqrt{1+\frac{2}{\beta}} &= 1+\frac{107}{114\beta}\\
1+\frac{2}{\beta} &= 1+\frac{107}{57\beta}+\frac{107^2}{114^2\beta^2}\\
\frac{107^2}{114^2\beta^2}-\frac{7}{57\beta} &= 0\\
\beta &= \frac{57}{7}\frac{107^2}{114^2}=\frac{11449}{1498}=7.17355889724
\end{aligned}$$

Using this, we calculate

$$\begin{aligned}
2s_0\left(\left(1+\frac{2}{\beta}\right)^{\frac{1}{2}}-1\right) &= 0.28l_0\\
\frac{s_0}{l_0} &= \frac{0.14}{(1+\frac{2}{7.17355889724})^{\frac{1}{2}}-1}=1.06999999996\\
\mathbb{E}(((X\wedge 10)-1)_+) &= 2s_0\left(\left(1+\frac{18}{7.17355889724}\right)^{\frac{1}{2}}-1\right)=1.74657965542s_0=1.86884023123l_0
\end{aligned}$$

So the ILF from \$500,000 to \$5,000,000 is 2.86884023123.