ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 3

Model Solutions

Basic Questions

1. *Loss amounts follow a gamma distribution with shape $\alpha = 2.4$ and scale $\theta = 500$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.880*
1	*0.074*
2	*0.035*
3	*0.011*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above $2,000. Calculate the expected payment for this excess-of-loss reinsurance.

If there are n claims, then the total losses follow a gamma distribution with shape $\alpha = 2.4n$ and $\theta = 500$. The expected payment on the excess of loss distribution in this case is therefore given by:

$$\begin{aligned}\mathbb{E}((X-2000)_+) &= \frac{500}{\Gamma(2.4n)}\int_{\frac{2000}{500}}^{\infty}\left(x-\frac{2000}{500}\right)x^{2.4n-1}e^{-x}\,dx\\ &= \frac{500}{\Gamma(2.4n)}\left(\int_4^{\infty}x^{2.4n}e^{-x}\,dx - 4\int_4^{\infty}x^{2.4n-1}e^{-x}\,dx\right)\\ &= 500\left(2.4n\texttt{pgamma(4,shape=2.4n+1,lower.tail=FALSE)}\right.\\ &\qquad\left.-4\texttt{pgamma(4,shape=2.4n,lower.tail=FALSE)}\right)\end{aligned}$$

This gives the following table

n	$P(N=n)$	$\mathbb{E}((S-2000)_+\vert N=n)$	$\mathbb{E}((S-2000)_+I_{N=n})$
0	0.930	0.00000	0.000000
1	0.074	91.92007	6.802085
2	0.035	635.21248	22.232437
3	0.011	1635.47244	17.990197

So the total expected payment is $6.802085 + 22.232437 + 17.990197 =$ \$47.02.

2. *Loss frequency follows a binomial distribution with $n = 17$ and $p = 0.36$. Loss severity (in thousands) has the following distribution:*

Severity	*Probability*
0	*0.31*
1	*0.23*
2	*0.11*
3	*0.18*
4 or more	*0.17*

Use the recursive method to calculate the exact probability that aggregate claims are at least \$4,000.

Recall that for the binomial distribution, $a = -\frac{p}{1-p} = -\frac{0.36}{0.64} = -\frac{9}{16}$ and $b = \frac{(n+1)p}{1-p} = \frac{18\times 0.36}{0.64} = \frac{81}{8}$. We compute $f_S(0) = P_S(0) = P_N(P_X(0)) = (1 - 0.36 + 0.36 f_X(0))^{17} = (1 - 0.36 + 0.36 \times 0.31)^{17} = 0.007794264046$

The recurrence formula is

$$f(x) = \frac{\sum_{k=1}^{x}\left(-\frac{9}{16} + \frac{153k}{16x}\right) f_X(k) f(x-k)}{1 + \frac{9}{16}\times 0.31} = \frac{\sum_{k=1}^{x}\frac{9}{16}\left(\frac{17k}{x} - 1\right) f_X(k) f(x-k)}{1.174375}$$

Applying this gives:

$$f(1) = \frac{\frac{9}{16}(18-1)\times 0.23 \times 0.007794264046}{1.174375} = 0.0145971342086$$

$$f(2) = \frac{\frac{9}{16}\left(\left(\frac{18}{2}-1\right)\times 0.23\times 0.0145971342086 + (18-1)\times 0.11 \times 0.007794264046\right)}{1.174375} = 0.0198459822453$$

$$f(3) = \frac{\frac{9}{16}\left(\left(\frac{18}{3}-1\right)\times 0.23\times 0.0198459822453 + \left(\frac{36}{3}-1\right)\times 0.11\times 0.0145971342086 + (18-1)\times 0.18\times 0.007794264046\right)}{1.174375} = 0.0308$$

The probability that aggregate claims are at least \$4,000 is therefore

$$\begin{aligned} & 1 - f(0) - f(1) - f(2) - f(3) \\ =& 1 - 0.007794264046 - 0.0145971342086 - 0.0198459822453 - 0.0308154731026 \\ =& 0.926947146397 \end{aligned}$$

3. *Use an arithmetic distribution ($h = 1$) to approximate a Gamma distribution distribution with shape $\alpha = 3$ and scale $\theta = \frac{5}{12}$.*

   (a) Using the method of rounding, calculate the mean of the arithmetic approximation. [You can evaluate this numerically: use 5,000 terms in the sum.]

Using the method of rounding, we set

$$
\begin{aligned}
p_0 &= P\left(X < \frac{1}{2}\right) \\
&= \int_0^{1.2} \frac{x^2 e^{-x}}{2}\, dx \\
&= \frac{1}{2}\left(\left[-x^2 e^{-x}\right]_0^{1.2} + \int_0^{1.2} 2xe^{-x}\, dx\right) \\
&= \frac{1}{2}\left(-1.44e^{-1.2} + \left[-2xe^{-x}\right]_0^{1.2} + \int_0^{1.2} 2e^{-x}\, dx\right) \\
&= \frac{1}{2}\left(-1.44e^{-1.2} - 2.4e^{-1.2} + 2.4(1 - e^{-1.2})\right) \\
&= \frac{1}{2}\left(2.4 - 6.24e^{-1.2}\right) \\
&= 0.260274058835
\end{aligned}
$$

and

$$
p_n = P\left(n - \frac{1}{2} \leqslant X < n + \frac{1}{2}\right)
$$

so $S_a(n) = S_x\left(n - \frac{1}{2}\right)$ and

$$
\mathbb{E}(X_a) = \sum_{n=1}^{\infty} S_X\left(n - \frac{1}{2}\right)
$$

We have

$$
\begin{aligned}
S_a(n) &= P\left(n - \frac{1}{2} \leqslant X\right) \\
&= \int_{2.4n-1.2}^{\infty} \frac{x^2 e^{-x}}{2}\, dx \\
&= \frac{1}{2}\left(\left[-x^2 e^{-x}\right]_{2.4n-1.2}^{\infty} + \int_{2.4n-1.2}^{\infty} 2xe^{-x}\, dx\right) \\
&= \frac{1}{2}\left((2.4n - 1.2)^2 e^{-(2.4n-1.2)} + \left[-2xe^{-x}\right]_{2.4n-1.2}^{\infty} + \int_{2.4n-1.2}^{\infty} 2e^{-x}\, dx\right) \\
&= \frac{1}{2}\left((2.4n - 1.2)^2 e^{-(2.4n-1.2)} + (4.8n - 2.4)e^{-(2.4n-1.2)} + 2.4(e^{-(2.4n-1.2)}\right) \\
&= \left(2.88n^2 - 0.48n + 0.52\right) e^{-(2.4n-1.2)}
\end{aligned}
$$

Thus

$$\begin{aligned}
\mathbb{E}(X_a) &= \sum_{n=1}^{\infty} \left(2.88n^2 - 0.48n + 0.52\right) e^{-(2.4n-1.2)} \\
&= e^{1.2} \left(2.88 \sum_{n=1}^{\infty} n^2 e^{-2.4n} - 0.48 \sum_{n=1}^{\infty} n^2 e^{-2.4n} + 0.52 \sum_{n=1}^{\infty} e^{-2.4n} \right)
\end{aligned}$$

From geometric series, we have $\sum_{n=1}^{\infty} e^{-cn} = \frac{e^{-c}}{1-e^{-c}}$, and we can calculate

$$\begin{aligned}
\sum_{n=1}^{\infty} ne^{-cn} &= \sum_{n=1}^{\infty} \left((n-1)+1\right) e^{-cn} \\
&= \sum_{m=0}^{\infty} me^{-c(m+1)} + \sum_{n=1}^{\infty} + \sum_{n=1}^{\infty} e^{-cn} \\
&= e^{-c} \sum_{m=0}^{\infty} me^{-cm} + \frac{e^{-c}}{1-e^{-c}} \\
(1-e^{-c}) \sum_{n=0}^{\infty} ne^{-cn} &= \frac{e^{-c}}{1-e^{-c}} \\
\sum_{n=0}^{\infty} ne^{-cn} &= \frac{e^{-c}}{(1-e^{-c})^2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 e^{-cn} &= \sum_{n=1}^{\infty} \left((n-1)^2 + 2n - 1\right) e^{-cn} \\
&= \sum_{m=0}^{\infty} m^2 e^{-c(m+1)} + \sum_{n=1}^{\infty} 2ne^{-cn} - \sum_{n=1}^{\infty} e^{-cn} \\
&= e^{-c} \sum_{m=0}^{\infty} m^2 e^{-cm} + \frac{2e^{-c}}{(1-e^{-c})^2} - \frac{e^{-c}}{1-e^{-c}} \\
(1-e^{-c}) \sum_{m=0}^{\infty} n^2 e^{-cn} &= \frac{e^{-c} + e^{-2c}}{(1-e^{-c})} \\
\sum_{m=0}^{\infty} n^2 e^{-cn} &= \frac{e^{-c} + e^{-2c}}{(1-e^{-c})^3}
\end{aligned}$$

Thus,

$$\mathbb{E}(X_a) = e^{1.2} \left(2.88 \frac{e^{-2.4} + e^{-4.8}}{(1-e^{-2.4})^3} - 0.48 \frac{e^{-2.4}}{(1-e^{-2.4})^2} + \frac{0.52e^{-2.4}}{1-e^{-2.4}} \right) = 1.25589406451$$

(b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value is larger than 6.5.

We have

$$1-(p_0+p_1+p_2+p_3+p_4+p_5+p_{6,l}) = S_X(6) = \frac{1}{2}\int_{6\times 2.4}^{\infty} x^2e^{-x}\,dx = e^{-14.4}\left(1+14.4+\frac{14.4^2}{2}\right) = 0.0000663740451726$$

and

$8.03583961878e-6$

$$\begin{aligned}
p_{6,u}+p_{7,l} &= \frac{1}{2}\left(\int_{6\times 2.4}^{\infty} x^2e^{-x}\,dx - \int_{7\times 2.4}^{\infty} x^2e^{-x}\,dx\right)\\
&= 0.0000663740451726 - e^{-16.8}\left(1+16.8+\frac{16.8^2}{2}\right)\\
&= 0.0000583382055538\\
6p_{6,u}+7p_{7,l} &= \frac{1}{2}\int_{14.4}^{16.8}\frac{5}{12}x^3e^{-x}\frac{d}{dx}\\
&= 1.25\left(e^{-14.4}\left(1+14.4+\frac{14.4^2}{2}+\frac{14.4^3}{6}\right) - e^{-16.8}\left(1+16.8+\frac{16.8^2}{2}+\frac{16.8^3}{6}\right)\right)\\
&= 0.000887312924604
\end{aligned}$$

So

$p_{6,u} = 7\times 0.0000583382055538 - 0.000369713718585 = 0.000038653720292$

Thus, $P(X_a > 6.5) = 0.0000663740451726 - 0.000038653720292 = 0.0000277203248806$.

Standard Questions

4. *The number of claims an insurance company receives follows a compound Poisson distribution with $\lambda = 2548$ for the primary distribution and $\lambda = 0.7$ for the secondary distribution. Claim severity follows a negative binomial distribution with $r = 0.2$ and $\beta = 12$. Calculate the probability that aggregate losses exceed \$6,000.*

   (a) Starting the recurrence 6 standard deviations below the mean [You need to calculate 15,000 terms of the recurrence for f_s.]

   We compute the intermediate distribution A by the usual recurrence, noting that $f_A(0) = P_A(0) = e^{0.7(f_X(0)-1)} = e^{0.7\left(13^{-0.2}-1\right)} = 0.755097800436$

and the recurrence is

$$f_A(x) = \sum_{k=1}^{x} 0.7\frac{k}{x} f_X(k) f_A(n-k)$$

We therefore compute the distribution of A using the usual recurrence:

```
fx<-choose(seq_len(40000)-0.8,seq_len(40000))*(12/13)^seq_len(40000)/13^0.2
### Since f_X(0) is not used in the recurrence, I have started the
### vector fx at f_X(1). This makes the indices slightly easier.

sum(fx)+1/13^0.2 # check we have enough terms

fA<-rep(exp(0.7*(13^(-0.2)-1)),40000)
for(i in seq_len(39999)){
    fA[i+1]<-0.7/i*sum(seq_len(i)*fx[seq_len(i)]*fA[i:1])
    ##Note that fA[i+1]=f_A(i), as this vector includes 0.
}
sum(fA) # check we have enough terms.
```

The mean and variance of A are given by the standard formulae:

$$\mathbb{E}(A) = 0.7 \times 0.2 \times 12 = 1.68$$
$$\mathrm{Var}(A) = 0.7 \times 0.2 \times 12 \times 13 + 0.7 \times (0.2 \times 12)^2 = 25.872$$

For the distribution of S, the recurrence is

$$f_S(x) = \sum_{k=1}^{x} 2548\frac{k}{x} f_X(k) f_A(n-k)$$

$f_S(0)$ is too small to start at zero. Therefore, we start the recurrence 6 standard deviations below the mean. The mean and standard deviation are given by

$$\mathbb{E}(S) = 2548 \times 1.68 = 4280.64$$
$$\mathrm{Var}(A) = 2548 \times 25.872 + 2548 \times 1.68^2 = 73113.3312$$

so 6 standard deviations below the mean is $4280.64 - 6\sqrt{73113.3312} = 2658.27138492$, so we start the recurrence from $f_S(2658) = 0$ and $f_S(2659) = 1$

```
fS<-rep(0,50000)
fS[2659]<-1 # Since we are truncating 0, we can let fS[1]=f_S(1)

for(i in seq_len(30000)){
    fS[2659+i]<-2548/(2659+i)*sum(seq_len(i)*fA[seq_len(i)+1]*fS[2658+(i:1)])
}

fS[32600] # check we have enough terms - this should be negligible.
fS<-fS/sum(fS) # rescale.
sum(fS[6001:32659]) #answer to question.
```

This gives the probability that $S > 6000$ as 4.03251×10^{-09}.

(b) Using a suitable convolution.

We can use the same code to get the distribution of f_A. Now we express S as a sum $S_1 + \cdots + S_8$, where S_i has a compound distribution with secondary distribution A, and primary distribution Poisson with mean 318.5. We compute the distribution of S_i using the standard recurrence

$$f_{S_i}(x) = \sum_{k=1}^{x} 318.5 \frac{k}{x} f_A(k) f_{S_i}(n-k)$$

with $f_{S_i}(0) = e^{318.5(f_A(0)-1)} = e^{318.5\times(0.755097800436-1)} = 1.33181490293 \times 10^{-34}$

```
fSi<-rep(exp(318.5*(fA[1]-1)),10001)
for(i in seq_len(10000)){ #10000 should be enough points
    fSi[i+1]<-318.5/i*sum(seq_len(i)*fA[seq_len(i)+1]*fSi[i:1])
}

sum(fSi8) # check that we have enough points

ConvolveSelf<-function(n){
    convolution<-vector("numeric",2*length(n))
    for(i in 1:(length(n))){
        convolution[i]<-sum(n[1:i]*n[i:1])
    }
    for(i in 1:(length(n))){
      convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)]*n[length(n)+1-(i:1)])
    }
    return(convolution)
}

### Convolve 8 times
fSi2<-ConvolveSelf(fSi)
fSi4<-ConvolveSelf(fSi2)
fSi8<-ConvolveSelf(fSi4)

sum(fSi8[6002:80000]) # gives the same answer as (a)
### Compare results in (a) and (b)
max(abs(fSi8[seq_len(35000)+1]-fS[seq_len(35000)]))
```

This gives the probability that $S > 6000$ as 4.106023×10^{-09}.

[The maximum difference in estimated probabilies between these two methods is 1.812565×10^{-12}.]