ACSC/STAT 4720, Life Contingencies II
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Homework Sheet 5
Model Solutions

Basic Questions

1. *An insurance company sells a 5-year annual life insurance policy to a life aged 41, for whom the lifetable below is appropriate.*

x	l_x	d_x
41	*10000.00*	*9.53*
42	*9990.47*	*11.05*
43	*9979.42*	*12.87*
44	*9966.56*	*15.05*
45	*9951.51*	*17.66*

The annual gross premium is \$385. Initial expenses are \$250 plus 25% of the first premium. The death benefits are \$230,000. Renewal costs are 3% of each subsequent premium. The interest rate is $i = 0.03$

(a) Calculate the expected net cash-flows associated with this policy (assuming no reserve). [This is the profit vector for the policy.]

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		346.25			−346.25
1	385	0.00	11.5500	219.19	177.36
2	385	11.55	11.2035	254.39	130.26
3	385	11.55	11.2035	296.62	88.03
4	385	11.55	11.2035	347.31	37.34
5	385	11.55	11.2035	408.16	−23.51

(b) Calculate the expected profit margin of the policy using a risk discount rate $i = 0.06$.

At $i = 0.06$, the NPV of the policy is

$$177.36(1.06)^{-1} + 0.999047 \times 130.26(1.06)^{-2} + 0.997942 \times 88.03(1.06)^{-3} + 0.996656 \times 37.34(1.06)^{-4} - 0.995151 \times 23.51(1.06)^{-5} - 346.25 = 22.6536$$

The NPV of premiums collected is

$$385+0.999047\times385(1.06)^{-1}+0.997942\times385(1.06)^{-2}+0.996656\times385(1.06)^{-3}+0.995151\times385(1.06)^{-4} = 1715.455$$

The profit margin is therefore $\frac{11.43049}{1715.455} = 0.01320559$.

2. *An insurance company sells a 5-year annual life insurance policy to a life aged 56, for whom the lifetable below is appropriate.*

x	l_x	d_x
56	*10000.00*	*11.06*
57	*9988.94*	*11.98*
58	*9976.96*	*13.02*
59	*9963.94*	*14.20*
60	*9949.74*	*15.52*

The annual gross premium is \$625. Initial expenses are \$130 plus 20% of the first premium. The death benefits are \$420,000. Renewal costs are 4% of each subsequent premium. The interest rate is $i = 0.03$. Reserves are calculated on the basis $i = 0.01$, with mortality following the table.

(a) Calculate the reserves.

Gross reserves:

The reserve in year 5 is $\frac{15.52}{9949.74} \times 420000(1.01)^{-1} - 0.96 \times 625 = 48.65$

The reserve in year 4 is $\frac{15.52(1.01)^{-2}+14.20(1.01)^{-1}}{9963.94} \times 420000 - 0.96 \times 625\left(1 + \frac{9949.74}{9963.94}(1.01)^{-1}\right) = 40.73$

The reserve in year 3 is $\frac{15.52(1.01)^{-3}+14.20(1.01)^{-2}+13.02(1.01)^{-1}}{9976.96} \times 420000 - 0.96 \times 625\left(1 + \frac{9963.94}{9976.96}(1.01)^{-1} + \frac{9949.74}{9976.96}(1.01)^{-2}\right) = -14.7483$ so no reserve is needed in year 3. Similarly, in earlier years, no reserve is needed.

Net reserves:

Premium is obtained by solving

$$P(1 + 0.998894(1.01)^{-1} + 0.997696(1.01)^{-2} + 0.996394(1.01)^{-3} + 0.994974(1.01)^{-4}) = 420000(0.001106(1.01)^{-1} + 0.00119$$
$$4.890282P = 2677.256$$
$$P = 547.46$$

This gives

t	Expected Death benefits	Expected premiums	Net reserves
2	2241.988	2153.47	88.52
3	1762.805	1624.01	138.79
4	1233.941	1088.74	145.20
5	648.646	547.46	101.18

(b) Calculate the profit signature.

Using Gross Reserves:

t	Initial Reserve	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Final Reserve	Net Cash Flow
0			255				−255
1	0.00	625	0	18.75	464.52	0.00	179.23
2	0.00	625	25	18.00	503.72	0.00	114.28
3	0.00	625	25	18.00	548.10	40.73	29.17
4	40.73	625	25	19.22	598.56	48.65	12.75
5	48.65	625	25	19.46	655.13	0.00	12.97

t	P(in force)	Pr_t	Π_t
0	1	-255.00	-255.00
1	1	179.23	179.23
2	0.998894	114.28	144.12
3	0.997696	29.17	29.10
4	0.996394	12.75	12.70
5	0.994974	12.97	12.90

Using Net Reserves:

t	Initial Reserve	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Final Reserve	Net Cash Flow
0			255				-255
1	0.00	625	25	18.00	464.52	88.42	90.81
2	88.52	625	25	20.66	503.72	138.62	66.79
3	138.79	625	25	22.16	548.10	145.01	67.84
4	145.20	625	25	22.36	598.56	101.04	67.96
5	101.18	625	25	21.03	655.13	0.00	67.09

t	P(in force)	Pr_t	Π_t
0	1	-255.00	-255.00
1	1	90.81	90.81
2	0.998894	66.79	66.71
3	0.997696	67.84	67.68
4	0.996394	67.96	67.71
5	0.994974	67.09	66.75

(c) Calculate the Discounted payback period at a risk rate of $i = 0.10$.

Using Gross Reserves:

At a risk rate of $i = 0.10$, the partial NPVs of the policy are

$$179.23(1.1)^{-1} - 255 = -92.06$$
$$179.23(1.1)^{-1} + 144.12(1.1)^{-2} - 255 = 27.04$$
$$179.23(1.1)^{-1} + 144.12(1.1)^{-2} + 29.10(1.1)^{-3} - 255 = 48.91$$
$$179.23(1.1)^{-1} + 144.12(1.1)^{-2} + 29.10(1.1)^{-3} + 12.70(1.1)^{-4} - 255 = 57.58$$
$$179.23(1.1)^{-1} + 144.12(1.1)^{-2} + 29.10(1.1)^{-3} + 12.70(1.1)^{-4} + 12.90(1.1)^{-5} - 255 = 65.59$$

So the discounted payback period is 2 years.

Using Net Reserves:

At a risk rate of $i = 0.10$, the partial NPVs of the policy are

$$
\begin{aligned}
90.81(1.1)^{-1} - 255 &= -172.45 \\
90.81(1.1)^{-1} + 66.79(1.1)^{-2} - 255 &= -117.25 \\
90.81(1.1)^{-1} + 66.79(1.1)^{-2} + 67.84(1.1)^{-3} - 255 &= -66.28 \\
90.81(1.1)^{-1} + 66.79(1.1)^{-2} + 67.84(1.1)^{-3} + 67.96(1.1)^{-4} - 255 &= -19.86 \\
90.81(1.1)^{-1} + 66.79(1.1)^{-2} + 67.84(1.1)^{-3} + 67.96(1.1)^{-4} + 67.09(1.1)^{-5} - 255 &= 21.80
\end{aligned}
$$

So the discounted payback period is 5 years.

3. *For the policy in Question 2:*

   (a) Calculate the reserves and profit signature for a general premium. [Hint: The reserves are positive starting in the third year.]

   If the premium is P, then the reserves are

$$
\begin{aligned}
2677.2561 - 4.694671P \\
2241.9883 - 3.776194P \\
1762.8053 - 2.847771P \\
1233.9408 - 1.909140P \\
648.6462 - 0.960000P
\end{aligned}
$$

   If the premium is at least 625, so that only the last two years' reserves are non-negative, then the profit signature is then given in the following table:

t	Initial Reserve	Prem. $(t-1)$	Expenses	Interest	Exp. D. Benefits	Final Reserve	Net Cash Flow
0			$130 + 0.2P$				$-130 - 0.2P$
1	0.00	P	$0.04P$	$0.0288P$	464.52	0.00	$0.9888P - 464.52$
2	0.00	P	$0.04P$	$0.0288P$	503.72	$1762.81 - 2.848P$	$3.836571P - 2266.52$
3	$1762.81 - 2.8478P$	P	$0.04P$	$52.88 - 0.0566P$	548.10	$1233.94 - 1.909P$	$33.65 + 0.0353P$
4	$1233.94 - 1.909P$	P	$0.04P$	$37.01 - 0.0261P$	598.56	$648.65 - 0.96P$	$23.75 + 0.0648P$
5	$648.65 - 0.96P$	P	$0.04P$	19.46	655.13	0.00	12.97

The profit signature is therefore:

$$
\begin{aligned}
& 0.9888P - 464.52 \\
0.998894(3.836571P - 2266.52) =& 3.832328P - 2264.02 \\
0.997696(33.65 + 0.0353P) =& 0.03521867P + 33.56 \\
0.996394(23.75 + 0.0648P) =& 0.06456627P + 23.66433 \\
0.994974 \times 12.97 =& 12.90772
\end{aligned}
$$

(b) Calculate the premium that gives an internal rate of return of $i = 0.15$.

At $i = 0.15$, the EPV of the policy is

$$(0.9888P-464.52)(1.15)^{-1}+(3.832328P-2264.02)(1.15)^{-2}+(0.03521867P+33.56)(1.15)^{-3}+(0.06456627P+23.66433)(1.15)^{-4}+(1$$

The premium which gives IRR of 0.15 is obtained by setting this equal to zero. That is $P = \frac{2328.83}{3.817689} = 610.01$.

We observe that this premium results in a positive reserve at the start of year 3 and a negative reserve at the start of years 1 and 2.

4. *For the policy in Question 1, use profit testing to calculate the reserves needed to ensure that all cash flows are non-negative.*

   Recall that the emerging cash flow in Year 5 of the policy is -23.51. The reserve needed to prevent a negative cashflow is therefore $23.51(1.03)^{-1} = 22.83$. This is less than the emerging cash-flow in year 4, so with the new reserve, there is still a positive cash-flow in earlier years. Therefore the only non-zero reserve is in year 5.

Standard Questions

5. *A couple purchase a reversionary annuity. Annual Premiums of \$14,830 are payable while both are alive. If the husband dies first, an annual life annuity of \$50,000 is payable to the wife until her death. The husband and wife are both aged 69 and both have mortality following the lifetable below. Assume both lives are independent.*

x	l_x	d_x
69	10000.00	99.59
70	9900.41	110.56
71	9789.85	122.59
72	9667.26	135.75
73	9531.51	150.11
74	9381.40	165.70
75	9215.70	182.57
76	9033.13	200.72
77	8832.41	220.14
78	8612.28	240.78
79	8371.50	262.54
80	8108.96	285.27
81	7823.69	308.76
82	7514.93	332.70
83	7182.23	356.71
84	6825.52	380.30
85	6445.22	402.88
86	6042.34	423.73
87	5618.61	442.05
88	5176.56	456.92
89	4719.64	467.38

x	l_x	d_x
90	4252.27	472.44
91	3779.83	471.15
92	3308.68	462.72
93	2845.96	446.54
94	2399.42	422.39
95	1977.03	390.48
96	1586.55	351.57
97	1234.97	307.05
98	927.93	258.85
99	669.08	209.41
100	459.68	161.42
101	298.26	117.51
102	180.75	79.90
103	100.85	50.02
104	50.83	28.29
105	22.55	14.08
106	8.47	5.93
107	2.54	1.99
108	0.54	0.48
109	0.06	0.06

Initial expenses are \$13,000, and renewal expenses are \$35 at the start of each subsequent year while the husband is alive, and \$20 at the start of each year while the husband is dead and the wife is alive. The interest

rate is $i = 0.05$. Use a profit test without reserves to determine the NPV of this policy at a risk discount rate of $i = 0.15$.

We first calculate the reserves needed while the husband is dead and the wife is alive:

t	Initial Reserve	Premium (at $t-1$)	Expenses	Annuity Payment	Interest	Expected Reserve
2	588630.18				26930.5090	565540.69
3	571927.53	0	20	50000	26095.3765	548002.91
4	554952.10	0	20	50000	25246.6050	530178.71
5	537729.64	0	20	50000	24385.4820	512095.12
6	520289.05	0	20	50000	23513.4525	493782.51
7	502660.81	0	20	50000	22632.0405	475272.85
8	484878.66	0	20	50000	21742.9330	456601.59
9	466978.05	0	20	50000	20847.9025	437805.44
10	448996.27	0	20	50000	19948.8135	418925.09
11	430974.16	0	20	50000	19047.7080	400001.87
12	412952.54	0	20	50000	18146.6270	381079.17
13	394974.20	0	20	50000	17247.7100	362201.91
14	377083.42	0	20	50000	16353.1710	343416.59
15	359324.56	0	20	50000	15465.2280	324769.78
16	341742.65	0	20	50000	14586.1325	306308.78
17	324382.52	0	20	50000	13718.1260	288080.65
18	307288.76	0	20	50000	12863.4380	270132.20
19	290504.34	0	20	50000	12024.2170	252508.55
20	274071.41	0	20	50000	11202.5705	235253.98
21	258029.50	0	20	50000	10400.4750	218409.46
22	242415.57	0	20	50000	9619.7785	202015.34
23	227265.19	0	20	50000	8862.2595	186107.45
24	212608.81	0	20	50000	8129.4405	170718.25
25	198475.05	0	20	50000	7422.7525	155877.81
26	184887.18	0	20	50000	6743.3590	141610.54
27	171865.46	0	20	50000	6092.2730	127937.73
28	159425.63	0	20	50000	5470.2815	114876.84
29	147579.60	0	20	50000	4877.9800	102436.47
30	136332.84	0	20	50000	4315.6420	90628.49
31	125690.34	0	20	50000	3783.5170	79452.12
32	115647.81	0	20	50000	3281.3905	68909.20
33	106203.25	0	20	50000	2809.1625	58992.41
34	97344.82	0	20	50000	2366.2410	49691.06
35	89059.58	0	20	50000	1951.9790	40991.56
36	81329.90	0	20	50000	1565.4950	32875.39
37	74137.37	0	20	50000	1205.8685	25323.24
38	67468.76	0	20	50000	872.4380	18321.20
39	61263.77	0	20	50000	562.1885	11805.96
40	55313.12	0	20	50000	264.6560	5557.78
41	50020.00	0	20	50000	0.0000	0.00

Now we can perform a profit test while the husband is alive: If the probability of the husband dying in year i

is q, then the probability of the husband dying and the wife surviving is $q(1-q)$.

We therefore get the following profit test in State 0.

t	Initial Reserve	Premium (at $t-1$)	Expenses	Interest Payment	Expected Reserve Alive	Expected Reserve Dead	Net cash Flow
1	0.00	14830	350	724.00	0.00	5803.79	9400.21
2	0.00	14830	350	724.00	0.00	6315.51	8888.49
3	0.00	14830	350	724.00	0.00	6862.18	8341.82
4	0.00	14830	350	724.00	0.00	7444.90	7759.10
5	0.00	14830	350	724.00	0.00	8064.89	7139.11
6	0.00	14830	350	724.00	0.00	8721.49	6482.51
7	0.00	14830	350	724.00	0.00	9415.52	5788.48
8	0.00	14830	350	724.00	0.00	10145.88	5058.12
9	0.00	14830	350	724.00	2912.12	10911.91	1379.97
10	3062.90	14830	350	877.14	6707.84	11712.20	0.00
11	7099.25	14830	350	1078.96	10113.69	12544.52	0.00
12	10779.18	14830	350	1262.96	13115.92	13406.21	0.00
13	14089.84	14830	350	1428.49	15704.12	14294.21	0.00
14	17021.07	14830	350	1575.05	17872.43	15203.69	0.00
15	19566.58	14830	350	1702.33	19619.01	16129.90	0.00
16	21723.22	14830	350	1810.16	20946.67	17066.72	0.00
17	23491.51	14830	350	1898.58	21862.64	18007.44	0.00
18	24875.27	14830	350	1967.76	22379.53	18943.51	0.00
19	25882.33	14830	350	2018.12	22514.08	19866.37	0.00
20	26523.41	14830	350	2050.17	22288.40	20765.19	0.00
21	26812.89	14830	350	2064.64	21728.62	21628.81	0.00
22	26767.65	14830	350	2062.38	20865.51	22444.51	0.00
23	26407.44	14830	350	2044.37	19733.80	23198.01	0.00
24	25754.06	14830	350	2011.70	18370.76	23875.00	0.00
25	24830.13	14830	350	1965.51	16817.93	24457.71	0.00
26	23660.15	14830	350	1907.01	15118.27	24928.89	0.00
27	22268.35	14830	350	1837.42	13317.00	25268.77	0.00
28	20678.80	14830	350	1757.94	11460.90	25456.02	0.00
29	18915.00	14830	350	1669.75	9595.82	25468.73	0.00
30	16997.05	14830	350	1573.85	7769.71	25281.20	0.00
31	14944.42	14830	350	1471.22	6028.29	24867.09	0.00
32	12771.97	14830	350	1362.60	4416.60	24197.97	0.00
33	10490.81	14830	350	1248.54	2977.22	23242.13	0.00
34	8106.72	14830	350	1129.34	1750.27	21965.78	0.00
35	5622.24	14830	350	1005.11	776.19	20331.16	0.00
36	3055.49	14830	350	876.77	115.10	18297.16	0.00
37	585.33	14830	350	753.27	0.00	15818.60	0.00
38	0.00	14830	350	724.00	0.00	12842.16	2361.84
39	0.00	14830	350	724.00	0.00	9286.11	5917.89
40	0.00	14830	350	724.00	0.00	4940.25	10263.75

At $i = 0.15$, this gives a NPV \$20359.19.

Alternative solution:

Taking the "without reserves" to mean with no reserves in either state, we can calculate the profit signature for each year by averaging over the possible states:

t	P(both alive)	P(H dead, W alive)	Exp. Prem.	Exp.	Exp. Ben.	Int.	Π_t
0				13000		−13000	
1	1.0000000	0.0000000	14830.00	0.00	0.00	741.50	15571.50
2	0.9801812	0.0098598	14536.09	343.26	492.99	684.99	14384.83
3	0.9584116	0.0205734	14213.24	335.86	1028.67	642.44	13491.16
4	0.9345592	0.0321668	13859.51	327.74	1608.34	596.17	12519.60
5	0.9084968	0.0446542	13473.01	318.87	2232.71	546.07	11467.50
6	0.8801067	0.0580333	13051.98	309.20	2901.67	492.06	10333.17
7	0.8492913	0.0722787	12594.99	298.70	3613.94	434.12	9116.47
8	0.8159744	0.0873386	12100.90	287.34	4366.93	372.33	7818.96
9	0.7801147	0.1031263	11569.10	275.10	5156.32	306.88	6444.57
10	0.7417137	0.1195143	10999.61	261.99	5975.72	238.10	5000.00
11	0.7008201	0.1363299	10393.16	248.01	6816.49	166.43	3495.09
12	0.6575523	0.1533437	9751.50	233.21	7667.18	92.56	1943.66
13	0.6121013	0.1702677	9077.46	217.64	8513.39	17.32	363.76
14	0.5647417	0.1867513	8375.12	201.39	9337.56	-58.19	-1222.03
15	0.5158443	0.2023787	7649.97	184.59	10118.94	-132.68	-2786.24
16	0.4658772	0.2166748	6908.96	167.39	10833.74	-204.61	-4296.78
17	0.4154086	0.2291134	6160.51	149.98	11455.67	-272.26	-5717.39
18	0.3650987	0.2391353	5414.41	132.57	11956.76	-333.75	-7008.66
19	0.3156878	0.2461732	4681.65	115.41	12308.66	-387.12	-8129.55
20	0.2679677	0.2496883	3973.96	98.78	12484.41	-430.46	-9039.70
21	0.2227500	0.2492140	3303.38	82.95	12460.70	-462.01	-9702.28
22	0.1808180	0.2444090	2681.53	68.17	12220.45	-480.35	-10087.45
23	0.1428711	0.2351119	2118.78	54.71	11755.59	-484.58	-10176.10
24	0.1094736	0.2213944	1623.49	42.74	11069.72	-474.45	-9963.42
25	0.0809949	0.2036011	1201.15	32.42	10180.06	-450.57	-9461.89
26	0.0575722	0.1823698	853.80	23.80	9118.49	-414.42	-8702.92
27	0.0390865	0.1586165	579.65	16.85	7930.83	-368.40	-7736.43
28	0.0251714	0.1334836	373.29	11.48	6674.18	-315.62	-6627.99
29	0.0152515	0.1082455	226.18	7.50	5412.27	-259.68	-5453.28
30	0.0086105	0.0841825	127.69	4.70	4209.12	-204.31	-4290.43
31	0.0044767	0.0624313	66.39	2.82	3121.57	-152.90	-3210.89
32	0.0021131	0.0438549	31.34	1.62	2192.75	-108.15	-2271.18
33	0.0008896	0.0289364	13.19	0.89	1446.82	-71.73	-1506.24
34	0.0003267	0.0177483	4.85	0.47	887.41	-44.15	-927.19
35	0.0001017	0.0099833	1.51	0.24	499.16	-24.89	-522.79
36	0.0000258	0.0050572	0.38	0.11	252.86	-12.63	-265.21
37	0.0000051	0.0022499	0.08	0.05	112.50	-5.62	-118.09
38	0.0000007	0.0008463	0.01	0.02	42.31	-2.12	-44.44
39	0.0000001	0.0002539	0.00	0.01	12.70	-0.64	-13.34
40	0.0000000	0.0000540	0.00	0.00	2.70	-0.14	-2.84
41	0.0000000	0.0000060	0.00	0.00	0.30	-0.02	-0.32

This gives a net present value of

$$15571.50(1.1)^{-1} + 14384.83(1.1)^{-2} + 13491.16(1.1)^{-3} + 12519.60(1.1)^{-4} + 11467.50(1.1)^{-5} + 10333.17(1.1)^{-6}$$
$$+9116.47(1.1)^{-7} + 7818.96(1.1)^{-8} + 6444.57(1.1)^{-9} + 5000.00(1.1)^{-10} + 3495.09(1.1)^{-11} + 1943.66(1.1)^{-12}$$
$$+363.76(1.1)^{-13} - 1222.03(1.1)^{-14} - 2786.24(1.1)^{-15} - 4296.78(1.1)^{-16} - 5717.39(1.1)^{-17} - 7008.66(1.1)^{-18}$$
$$-8129.55(1.1)^{-19} - 9039.70(1.1)^{-20} - 9702.28(1.1)^{-21} - 10087.45(1.1)^{-22} - 10176.10(1.1)^{-23} - 9963.42(1.1)^{-24}$$
$$-9461.89(1.1)^{-25} - 8702.92(1.1)^{-26} - 7736.43(1.1)^{-27} - 6627.99(1.1)^{-28} - 5453.28(1.1)^{-29} - 4290.43(1.1)^{-30}$$
$$-3210.89(1.1)^{-31} - 2271.18(1.1)^{-32} - 1506.24(1.1)^{-33} - 927.19(1.1)^{-34} - 522.79(1.1)^{-35} - 265.21(1.1)^{-36}$$
$$-118.09(1.1)^{-37} - 44.44(1.1)^{-38} - 13.34(1.1)^{-39} - 2.84(1.1)^{-40} - 0.32(1.1)^{-41} - 13000 = \$44,282.94$$