ACSC/STAT 4720, Life Contingencies II
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Homework Sheet 3
Model Solutions

Basic Questions

1. *An individual aged 39 has a current salary of \$104,000. The salary scale is $s_y = 1.05^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

   The final 3 years are the years at the start of which the individual is aged 62, 63 and 64. The final average salary is therefore

   $$104000\frac{(1.05)^{23}+(1.05)^{24}+(1.05)^{25}}{3} = \$335,676.59$$

2. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 60% of final average salary for an employee who enters the plan at exact age 32, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 70% of the life annuity.*
   - *At age 65, the employee is married to someone aged 60.*
   - *The salary scale is $s_y = 1.04^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000016(1.091)^x$.*
   - *A fixed percentage of salary is payable monthly in arrear.*
   - *Contributions earn an annual rate of 6%.*
   - *The value of the life annuity is based on $\delta = 0.05$.*

   Calculate the percentage of salary payable monthly to achieve the target replacement rate under these assumptions. [You may use numerical integration to compute the value of the annuities.]

   The survival probability for an individual aged 65 is

   $$e^{-\int_{65}^{65+t}0.0000016(1.091)^x\,dx} = e^{-\int_{65}^{65+t}0.0000016e^{x\log(1.091)}\,dx} = e^{-\left[\frac{0.0000016}{\log(1.091)}e^{x\log(1.091)}\right]_{65}^{65+t}} = e^{-0.005281253\left((1.091)^t-1\right)}$$

   We have

   $$\overline{a}_{65} = \int_0^\infty e^{-0.05t}e^{-0.005281253\left((1.091)^t-1\right)}\,dt$$

   Evaluating this numerically gives

   $$\overline{a}_{65} = 18.1875$$

   Similarly the survival probability for an individual aged 60 is

   $$e^{-\int_{60}^{60+t}0.0000016(1.091)^x\,dx} = e^{-\int_{60}^{60+t}0.0000016e^{x\log(1.091)}\,dx} = e^{-\left[\frac{0.0000016}{\log(1.091)}e^{x\log(1.091)}\right]_{60}^{60+t}} = e^{-0.00341675\left((1.091)^t-1\right)}$$

The probability that t years after retirement the individual is dead and the spouse is alive is therefore

$$e^{-0.00341675\left((1.091)^t-1\right)}\left(1-e^{-0.005281253\left((1.091)^t-1\right)}\right)$$

$$\overline{a}_{65|60}=\int_0^\infty e^{-0.05t}e^{-0.00341675\left((1.091)^t-1\right)}\left(1-e^{-0.005281253\left((1.091)^t-1\right)}\right)\,dt$$

Evaluating this numerically gives

$$\overline{a}_{65|60}=0.9019536$$

If the final average salary is F, then the cost of these annuities is $(18.1875\times 0.6+0.9019536\times 0.6\times 0.7)F=11.29132F$.

If the employee starts with salary S at age 32, then the employee's final average salary is $F=\frac{1.04^{30}+1.04^{31}+1.04^{32}}{3}S=3.374863S$.

The accumulated value should therefore be $11.29132\times 3.374863S=38.10666S$. Let M be the employee's first month's salary. If the employee contributed all their salary into the pension plan, the accumulated value would be $M\frac{1.06^{33}-1.04^{33}}{1.06^{\frac{1}{12}}-1.04^{\frac{1}{12}}}=2002.878M$. We also have that the first year's salary is $S=\frac{1.04-1}{1.04^{\frac{1}{12}}-1}M=12.21844M$. Therefore, the accumulated value of the pension fund would be $\frac{2002.878}{12.21844}S=163.9226S$

The proportion of salary payable to the pension fund is therefore

$$\frac{38.10666}{163.9226}=23.25\%$$

3. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	1.000000	39	1.350398	48	1.845766	57	2.553877
31	1.033333	40	1.397268	49	1.912422	58	2.649694
32	1.067933	41	1.445983	50	1.981785	59	2.749515
33	1.103853	42	1.496620	51	2.053975	60	2.853522
34	1.141149	43	1.549263	52	2.129115	61	2.961903
35	1.179879	44	1.604000	53	2.207337	62	3.074855
36	1.220103	45	1.660921	54	2.288777	63	3.192585
37	1.261887	46	1.720122	55	2.373580	64	3.315310
38	1.305295	47	1.781702	56	2.461894	65	3.443256

An employee aged 34 and 7 months has 11 years of service, and a current salary of \$62,000 (for the coming year). She has a defined benefit pension plan with $\alpha=0.02$ and S_{Fin} is the average of her last 3 years' salary. The employee's mortality is given by $\mu_x=0.00000224(1.101)^x$. The pension benefit is payable monthly in advance. The interest rate is $i=0.04$. [This gives $\ddot{a}_{65}^{(12)}=19.62939$.] Calculate the EPV of the accrued benefit under the assumption that the employee retires at age 65.

Using linear interpolation, we get

$$s_{34\frac{7}{12}}=\frac{7}{12}s_{35}+\frac{5}{12}s_{34}=\frac{7\times 1.179879+5\times 1.141149}{12}=1.163741$$

The estimated final average salary for this employee is therefore

$$\frac{3.074855+3.192585+3.315310}{3\times 1.163741}\times 62000=\$170178.33$$

The accrued annual annuity payment is therefore $170178.33 \times 0.02 \times 11 = \$37,439.23$.

The value of the pension at time of retirement is therefore

$$19.62939 \times 37439.23 = \$734,909.25$$

We need to discount this by 30 years 5 months, so the EPV is

$$734909.25(1.04)^{-30\frac{5}{12}} = \$222,913.47$$

Standard Questions

4. *An employee aged 58 has been working with a company for 28 years. The employee's salary last year was \$58,000. The salary scale is the same as for Question 3. The service table is given below:*

t	$_tp^{(00)}$	1	2	3
0	10000.00	48.48	0	6.48
1	9945.03	45.29	0	7.29
2^-	9892.45		1137.53	
2	8754.92	25.72	142.14	6.86
3	8580.20	21.69	128.54	7.56
4^-	8422.40		1426.64	
4	6995.76	12.81	455.53	6.95
5	6520.47	10.83	757.69	7.29
6	5744.66	9.16	416.44	7.23
7^-	5311.82		5311.82	

Mortality follows a Gompertz model with $B = 0.0000012$ and $C = 1.1$. The member's current salary is \$74,000. If the member withdraws before age 60, he receives a defered pension starting from age 65, with 2% COLA. The death benefit of the plan is three times the employee's final average salary if the employee is still working at the time of death. If the employee has withdrawn, the death benefit is three times final average salary with COLA of 3%. The accrual rate for the pension is 0.02. Pension payments are made annually in advance. The interest rate is $i = 0.06$.

Calculate the EPV of the accrued benefit. [You may assume that events happen in the middle of each year.]

We have the following values:

x	a_x
60	16.52493
60.5	16.49421
61.5	16.43047
62	16.39741
62.5	16.36354
63.5	16.29327
64.5	16.21955
65	16.18134

For an individual who withdraws at age x with final average salary F, the expected death benefit paid during the withdrawl period is

$$3F\int_0^{65-x}\mu_{x+t}e^{-\int_0^t\mu_{x+s}\,ds}(1.02)^t(1.06)^{-t}\,dt = 3F\int_0^{65-x}0.0000012(1.1)^{x+t}e^{-\frac{0.0000012}{\log(1.1)}(1.1^{x+t}-1.1^x)}(1.02)^t(1.06)^{-t}\,dt$$
$$= 3F0.0000012(1.1)^x\int_0^{65-x}e^{-\frac{0.0000012(1.1)^x}{\log(1.1)}(1.1^t-1)}\left(\frac{1.122}{1.06}\right)^t\,dt$$

We calculate the following:

Age at exit	Final ave. sal.	COLA to age 65	Exp. D. Ben.	Exp Def pension	Exp Pen Ben	Exp W. D. Ben.	Withdrawn Surv prob	Exp Tot Ben	Exp Disc Tot. Ben.
58.5	61312.40	1.137368	119.19	2091.72	0.00	5.47	0.9971529	2216.38	2152.74
59.5	63608.27	1.115067	139.11	2107.52	0.00	4.87	0.9974843	2251.51	2063.07
60	64779.64	1.104081	0.00	0.00	68191.31	0.00	0.9976623	68191.31	60690.03
60.5	65999.98	1.093203	135.83	1291.07	8665.22	2.56	0.9978490	10094.68	8726.26
61.5	68491.82	1.071768	155.34	1174.67	8100.56	1.91	0.9982503	9432.48	7692.29
62	69763.33	1.061208	0.00	0.00	91391.30	0.00	0.9984658	91391.30	72390.47
62.5	71088.31	1.050752	148.22	748.77	29674.30	0.92	0.9986919	30572.21	23520.71
63.5	73794.17	1.030150	161.39	683.30	51016.41	0.54	0.9991779	51861.64	37641.24
64.5	76614.33	1.009950	166.18	623.83	28979.39	0.19	0.9997127	29769.59	20383.78
65	78053.62	1.000000	0.00	0.00	375698.00	0.00	1.0000000	375698.00	249860.63

So the total EPV of the accrued benefit is $485,121.21.

5. *An individual aged 39 has 13 years of service, and last year's salary was $204,000. The salary scale is $s_y = 1.04^y$. The accrual rate is 0.015. The interest rate is $i = 0.03$. There is no death benefit, and no exits other than death or retirement at age 65. Mortality follows a Gompertz law with $B = 0.000002$ and $C = 1.08$. Pension benefits are payable annually in advance. You are given that $\ddot{a}_{65} = 21.85281$. Calculate this year's employer contribution to the plan using*

   (a) The projected unit method.

   Under the projected unit method, the estimated final average salary is $204000\frac{1.04^{23}+1.04^{24}+1.04^{25}}{3} = \$523,182.21$. If the employee retires at age 65, the EPV of the accrued pension at that time is therefore

$$21.85281\times 523,182.21\times 0.015\times 13 = \$2,229,435.28$$

   The current EPV is therefore

$$2229435.28(1.03)^{-26}e^{-\frac{0.000002}{\log(1.08)}(1.08^{65}-1.08^{39})} = \$1,030,326.59$$

   In one year's time, the current EPV of the pension will be

$$21.85281\times 523,182.21\times 0.015\times 14(1.03)^{-25}e^{-\frac{0.000002}{\log(1.08)}(1.08^{65}-1.08^{40})} = \$1,142,917.75$$

   The expected present value of this is

$$1142917.75(1.03)^{-1}e^{-\frac{0.000002}{\log(1.08)}(1.08^{65}-1.08^{40})} = 1,109,582.48$$

The employer contribution is therefore

$$1109582.48 - 1030326.59 = \$79,255.89$$

(b) The traditional unit method.

Using the traditional unit method the current final average salary is $204000\frac{1.04^{-3}+1.04^{-2}+1.04^{-1}}{3} = \$188,706.19$. The current EPV is therefore

$$21.85281 \times 188706.19 \times 0.015 \times 13(1.03)^{-26}e^{-\frac{0.000002}{\log(1.08)}(1.08^{65}-1.08^{39})} = \$371627.71$$

After another year, the employee's final average salary will be $204000\frac{1.04^{-2}+1.04^{-1}+1}{3} = \$196,254.44$.
The current EPV of the accrued benefits at this point is therefore

$$21.85281 \times 196254.44 \times 0.015 \times 13(1.03)^{-26}e^{-\frac{0.000002}{\log(1.08)}(1.08^{65}-1.08^{39})} = \$416,223.03$$

This year's employer contribution is the change in EPV, which is $416223.03 - 371627.71 = \$44,595.32$