ACSC/STAT 4720, Life Contingencies II
Fall 2018

Toby Kenney

Homework Sheet 6

Model Solutions

Basic Questions

1. *An individual aged 35 has a current salary of $83,000. The salary scale is $s_y = 1.04^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

   The individual's last 3 years start at ages 62, 63 and 64, which are respectively 27, 28 and 29 years in the future. The final average salary is therefore

$$83000\left(\frac{1.04^{27}+1.04^{28}+1.04^{29}}{3}\right) = \$249,020.01$$

2. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 75% of final average salary for an employee who enters the plan at exact age 29, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 70% of the life annuity.*
   - *At age 65, the employee is married to someone aged 60.*
   - *The salary scale is $s_y = 1.04^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000013(1.102)^x$.*
   - *A fixed percentage of salary is payable monthly in arrear.*
   - *Contributions earn an annual rate of 7%.*
   - *The value of the life annuity is based on $\delta = 0.06$.*

   Calculate the percentage of salary payable monthly to achieve the target replacement rate under these assumptions. [You may use numerical integration to compute the value of the annuities.]

   Using numerical integration, we get that

$$\overline{a}_{65} = \int_0^\infty e^{\frac{0.0000013}{\log(1.102)}(1.102^{65}-1.102^{65+t})}e^{-0.06t}\,dt = 15.10939$$

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sum(exp(0.0000013*(1.102^65-1.102^(65+(0:100000)/500))-0.06*(0:100000)/500)))/500
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   Similarly, we get that

$$\overline{a}_{65|60} = \int_0^\infty \left(1-e^{\frac{0.0000013}{\log(1.102)}(1.102^{65}-1.102^{65+t})}\right)e^{\frac{0.0000013}{\log(1.102)}(1.102^{60}-1.102^{60+t})}e^{-0.06t}\,dt = 0.8259532$$

If the initial salary is S, then the employee's projected final average salary is

$$S\left(\frac{1.04^{33}+1.04^{34}+1.04^{35}}{3}\right)=3.79626214383S$$

The expected cost of the annuities at time of retirement is therefore

$$3.79626214383\times 0.75\times(15.10939+0.7\times 0.8259532)S=44.6655597595S$$

Making monthly contributions of $\frac{1}{12}$ in arrear, if the first monthly salary is M, then the accumulated value of this individual investing her whole salary in the plan is

$$\frac{1.07^{36}-1.04^{36}}{1.07^{\frac{1}{12}}-1.04^{\frac{1}{12}}}M=3075.1102265M$$

Her initial annual salary is $S=\frac{1.04-1}{1.04^{\frac{1}{12}}-1}M=12.2184421145M$. Therefore if she invests her whole salary in the plan, the accumulated value is $\frac{3075.1102265}{12.2184421145}S=251.67776691S$. The percentage of salary she needs to invest is therefore $\frac{44.6655597595}{251.67776691}=17.747\%$.

3. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	1.000000	39	1.350398	48	1.845766	57	2.553877
31	1.033333	40	1.397268	49	1.912422	58	2.649694
32	1.067933	41	1.445983	50	1.981785	59	2.749515
33	1.103853	42	1.496620	51	2.053975	60	2.853522
34	1.141149	43	1.549263	52	2.129115	61	2.961903
35	1.179879	44	1.604000	53	2.207337	62	3.074855
36	1.220103	45	1.660921	54	2.288777	63	3.192585
37	1.261887	46	1.720122	55	2.373580	64	3.315310
38	1.305295	47	1.781702	56	2.461894	65	3.443256

An employee aged 41 and 5 months has 6 years of service, and a current salary of $44,000 (for the coming year). He has a defined benefit pension plan with $\alpha=0.02$ and S_{Fin} is the average of his last 3 years' salary. The employee's mortality is given by $\mu_x=0.0000012(1.112)^x$. The pension benefit is payable monthly in advance. The interest rate is $i=0.05$. [This gives $\ddot{a}_{65}^{(12)}=19.05482$.] Calculate the EPV of the accrued benefit under the assumption that the employee retires at age 65.

We compute

$$s_{41\frac{5}{12}}=\frac{5}{12}s_{42}+\frac{7}{12}s_{41}=\frac{5}{12}\times 1.496620+\frac{7}{12}\times 1.445983=1.46708175$$

Therefore the final average salary is

$$44000\times\frac{3.074855+3.192585+3.315310}{3\times 1.46708175}=95800.3874018$$

The EPV of the accrued benefit is therefore

$$95800.3874018\times 0.02\times 6\times 19.05482\times(1.05)^{-23-\frac{7}{12}}=69316.8883456$$

Standard Questions

4. *An employee aged 57 has been working with a company for 24 years. The employee's salary last year was \$59,000. The salary scale is the same as for Question 3. The service table is given below:*

t	$_tp^{(00)}$	1	2	3
0	10000.00	46.08	0	6.68
1	9947.24	45.46	0	7.09
2	9894.69	45.10	0	7.42
2^-	9842.17		1064.02	
2	8778.15	28.23	163.43	6.96
3	8579.53	26.39	183.80	7.24
4^-	8362.10		1316.65	
4	7045.45	17.26	334.03	6.88
5	6687.28	15.33	377.95	7.19
6	6286.81	4.66	426.03	7.53
7^-	5848.59		5848.59	

Mortality follows a Gompertz model with $B = 0.0000015$ and $C = 1.107$. If the member withdraws, she receives a defered monthly pension starting from age 65, with 2% COLA. The death benefit of the plan is three times the employee's final average salary if the employee is still working at the time of death. If the employee has withdrawn, the death benefit is three times final average salary with COLA of 2%. The accrual rate for the pension is 0.015. Pension payments are made annually in advance. The interest rate is $i = 0.05$.

Calculate the EPV of the accrued benefit. [You may assume that events happen in the middle of each year.]

You are given the following values:

x	$\ddot{a}_x^{(12)}$
60	19.51542
60.5	19.49118
61.5	19.44087
62	19.41477
62.5	19.38803
63.5	19.3326
64.5	19.27451
65	19.24444

We compute values for each age:

For example, the EPV of accrued pension benefits for individuals retiring at age 60 are given by:

$$63532.6668545 \times 0.36 \times 0.106402 \times (1.05)^{-3} \times 19.51542$$

The EPV of accrued deferred pension benefits for someone withdrawing at age 57.5 are given by

$$57970.443217 \times 1.02^{7.5} \times 0.36 \times 19.24444 \times 0.004608 \times e^{-\frac{0.0000015}{\log(1.107)}\left((1.107)^{65}-(1.107)^{57.5}\right)} \times (1.05)^{-8}$$

The EPV of death benefits in deferment for someone withdrawing at age 57.5 are given by

$$57970.443217\times\times 3\times 0.004608\times\int_0^{7.5} e^{-\frac{0.0000015}{\log(1.107)}\left((1.107)^{57+t}-(1.107)^{57.5}\right)}\times 0.0000015(1.107)^{57.5+t}\times(1.05)^{-(t+0.5)}\times(1.02)^t\,dt$$

The EPV of death benefits for someone dying at age 57.5 are

$$57970.443217\times 3\times 0.000668\times(1.05)^{-0.5}$$

Age	Final Average Salary	P(survive to 65)	EPV Deferred Pension Benefits	EPV Deferred Death Benefits	EPV Death Benefits	EPV Pension Benefits
57.5	57970.443217	0.994187048662	1444.71	4.68	113.37	
58.5	60132.16816	0.994729582484	1450.23	4.12	118.87	
59.5	62383.8468135	0.995330512708	1464.23	3.57	122.92	
60	63532.6668545	0.995654827245	0	0	0	41026.02
60.5	64729.513334	0.995996165851	932.96	1.89	113.94	6257.66
61.5	67173.389607	0.996733563063	887.99	1.42	117.14	6937.60
62	68420.419401	0.997131568836	0	0	0	49333.77
62.5	69719.8972285	0.997550498484	591.46	0.68	110.03	12429.03
63.5	72373.665492	0.998455626886	535.11	0.38	113.68	13863.64
64.5	75139.539382	0.999458561843	165.73	0.04	117.72	15405.47
65	76551.122835	1	0	0	0	209940.45
Total			7472.44	16.78	927.69	355193.64

The accrued benefit is therefore $7472.43979218+16.7798892452+927.68869161+355193.640836=\$363,610.55$.

5. *An individual aged 53 has 11 years of service, and last year's salary was \$87,000. The salary scale is $s_y=1.04^y$. The accrual rate is 0.015. The interest rate is $i=0.04$. There is no death benefit, and no exits other than death or retirement at age 65. The pension benefit is payable annually in advance. Mortality follows a Gompertz law with $B=0.0000034$ and $C=1.14$. You are given that $\ddot{a}_{65}^{(12)}=20.9262$. Calculate this year's employer contribution to the plan using*

   (a) The projected unit method.

   The probability of the employee surviving to age 65 at the start of the year is $e^{-\frac{0.0000034}{\log(1.14)}\left((1.14)^{65}-(1.14)^{53}\right)}=0.902328123975$. The probability of surviving to age 65 from the end of the year is $e^{-\frac{0.0000034}{\log(1.14)}\left((1.14)^{65}-(1.14)^{54}\right)}=0.9057352031$. The individual's projected final average salary is $87000\frac{1.04^{10}+1.04^{11}+1.04^{12}}{3}=134001.186235$. The current accrued pension benefit is therefore

$$134001.186235\times 11\times 0.015\times 20.9262\times 0.902328123975\times(1.04)^{-12}=260763.847952$$

   The accrued pension benefit if the life survives to the end of the year is

$$134001.186235\times 12\times 0.015\times 20.9262\times 0.9057352031\times(1.04)^{-11}=296965.525429$$

   The expected accrued value is

$$296965.525429\times\frac{0.902328123975}{0.9057352031}(1.04)^{-1}=284469.652312$$

The normal contribution is therefore $284469.652312 - 260763.847952 = \$23,705.80$.

(b) The traditional unit method.

The individual's current final average salary is $87000\frac{1.04^{-2}+1.04^{-1}+1}{3} = 83696.7455623$. The current accrued pension benefit is therefore

$$83696.7455623 \times 11 \times 0.015 \times 20.9262 \times 0.902328123975 \times (1.04)^{-12} = 162872.330067$$

Next year's final average salary is $87000\frac{1.04^{-1}+1+1.04}{3} = 87044.6153844$ The accrued pension benefit if the life survives to the end of the year is

$$87044.6153844 \times 12 \times 0.015 \times 20.9262 \times 0.9057352031 \times (1.04)^{-11} = 192903.142649$$

The expected accrued value is

$$192903.142649 \times \frac{0.902328123975}{0.9057352031}(1.04)^{-1} = 184786.061749$$

The normal contribution is therefore $184786.061749 - 162872.330067 = \$21,913.73$.