ACSC/STAT 4720, Life Contingencies II
Fall 2018
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Homework Sheet 7
Model Solutions

Basic Questions

1. *An insurance company sells a 5-year annual life insurance policy to a life aged 36, for whom the lifetable below is appropriate.*

x	l_x	d_x
36	*10000.00*	*9.64*
37	*9990.36*	*10.13*
38	*9980.23*	*10.68*
39	*9969.55*	*11.30*
40	*9958.25*	*11.98*

The annual gross premium is \$625.40. Initial expenses are \$190 plus 40% of the first premium. The death benefits are \$500,000. Renewal costs are 3% of each subsequent premium. The interest rate is $i = 0.04$

(a) Calculate the expected net cash-flows associated with this policy (assuming no reserve). [This is the profit vector for the policy.]

We conduct the following profit test:

Year	Premium	Expenses	Interest	Expected Death Benefits	$\Pr_t$
0		440.16			−440.16
1	625.40	0	25.016	482	168.416
2	625.40	18.762	24.26552	506.988737143	123.914782857
3	625.40	18.762	24.26552	535.057809289	95.845710711
4	625.40	18.762	24.26552	566.725679695	64.177840305
5	625.40	18.762	24.26552	601.511309718	29.392210282

(b) Which of the following is the internal rate of return of the policy:

(i) $i = 0.0940$

(ii) $i = 0.1244$

(iii) $i = 0.1576$

(iv) $i = 0.1694$

We first compute the profit signature

Year	$\Pr_t$	P(in force)	Π_t
0	−440.16	1	−440.16
1	168.416	1	168.416
2	123.914782857	0.999036	123.795329006
3	95.845710711	0.998023	95.6562237409
4	64.177840305	0.996955	63.9824187813
5	29.392210282	0.995825	29.2694978041

The NPV at interest rate i is therefore

$$168.416(1+i)^{-1}+123.795329006(1+i)^{-2}+95.6562237409(1+i)^{-3}+63.9824187813(1+i)^{-4}+29.2694978041(1+i)^{-5}-440.16$$

Substituting the given values of i we get:

i	NPV
0.04002	0.00312808
(i) 0.0940	−46.377043459
(ii) 0.1244	−68.853852347
(iii) 0.1576	−90.914432701
(iv) 0.1694	−98.198168206

We see that none of the values given is the true rate of return, which is $i = 0.04002$.

2. *An insurance company sells a 5-year annual life insurance policy to a life aged 40, for whom the lifetable below is appropriate.*

x	l_x	d_x
40	*10000.00*	*6.63*
41	*9993.37*	*7.37*
42	*9986.00*	*8.20*
43	*9977.80*	*9.12*
44	*9968.68*	*10.16*

The annual gross premium is \$280. Initial expenses are \$80 plus 25% of the first premium. The death benefits are \$300,000. Renewal costs are 2% of each subsequent premium. The interest rate is $i = 0.05$. Reserves are calculated on the basis $i = 0.03$, with mortality following the table.

(a) Calculate the reserves.

On the reserve basis, we compute:

Year	$\ddot{a}_{t:\overline{5-t}\rceil}$	$A^1_{t:\overline{5-t}\rceil}$	Reserve
	4.7103207233	0.00377550937872	0
	3.8241657669	0.00322791476757	0
	2.91103759799	0.00258917274172	0
	1.96998637947	0.00184721515452	13.600283849
	1	0.000989506902609	22.452070783

(b) Calculate the profit signature.

We conduct the following profit test:

Year	Initial Reserves	Premium	Expenses	Interest	Expected Death Benefits	Expected Res. Payments	$\Pr_t$
0			150				−150
1	0	280	0	14	198.9	0	95.1
2	0	280	5.4	13.73	221.246686553	0	67.083313447
3	0	280	5.4	13.73	246.344882836	13.5891159812	28.3960011828
4	13.600283849	280	5.4	14.4100141925	274.20874341	22.4315489359	5.9700056961
5	22.452070783	280	5.4	14.8526035392	305.757632906	0	6.147041416

The profit signature is then calculated as follows:

Year	Pr_t	P(in force)	Π_t
0	-150	1	-150
1	95.1	1	95.1
2	67.083313447	0.999337	67.0388372102
3	28.3960011828	0.998600	28.3562467811
4	5.9700056961	0.997780	5.95675228345
5	6.147041416	0.996868	6.12778888229

(c) Calculate the profit margin at a risk discount rate of $i = 0.08$.

At risk discount rate $i = 0.08$, the NPV is

$$95.1(1.08)^{-1} + 67.04(1.08)^{-2} + 28.36(1.08)^{-3} + 5.96(1.08)^{-4} + 6.13(1.08)^{-5} - 150 = 26.589516995$$

The EPV of premiums received is

$$280(1 + 0.999337(1.08)^{-1} + 0.998600(1.08)^{-2} + 0.997780(1.08)^{-3} + 0.996868(1.08)^{-4}) = 1205.74951161$$

The profit margin is therefore $\frac{26.589516995}{1205.74951161} = 2.2052\%$.

3. For the policy in Question 2:

(a) Calculate the reserves and profit signature for a general premium. [You may assume that P is such that the reserves are zero in Years 1, 2 and 3.]

The reserves in Years 4 and 5 are $554.164546356 - 1.93058665188P$ and $296.852070783 - 0.98P$ respectively. The profit test for premium P becomes

Year	Initial Reserves	Premium	Expenses	Interest	Expected Death Benefits	Expected Res. Payments	Pr_t
0			$80 + 0.25P$				$-80 - 0.25P$
1	0	P	0	$0.05P$	198.9	0	$1.05P - 198.9$
2	0	P	$0.02P$	$0.049P$	221.246686553	0	$1.029P - 221.25$
3	0	P	$0.02P$	$0.049P$	246.344882836	$553.71 - 1.9290P$	$2.9580P - 800.05$
4	$554.16 - 1.9306P$	P	$0.02P$	$27.708 - 0.0475P$	274.20874341	$296.58 - 0.9791P$	$11.08 - 0.0190P$
5	$296.85 - 0.98P$	P	$0.02P$	14.843	305.757632906	0	5.937041416

The profit signature is then calculated as follows:

Year	Pr_t	P(in force)	Π_t
0	$-80 - 0.25P$	1	$-80 - 0.25P$
1	$1.05P - 198.9$	1	$1.05P - 198.9$
2	$1.029P - 221.246686553$	0.999337	$1.028317773P - 221.1$
3	$2.9580013514P - 800.054377191$	0.998600	$2.95386014951P - 798.934301063$
4	$11.083290926 - 0.019011733036P$	0.997780	$11.0586860201 - 0.0189695269887P$
5	5.937041416	0.996868	5.91844660229

(b) Calculate the premium that gives an internal rate of return of $i = 0.14$.

At risk discount rate $i = 0.14$, the NPV is

$$(1.05P - 198.9)(1.14)^{-1} + (1.0283P - 221.1)(1.14)^{-2} + (2.9539P - 798.93)(1.14)^{-3} + (11.06 - 0.01897P)(1.14)^{-4}$$
$$+ (5.92)(1.14)^{-5} - (80 + 0.25P)$$
$$= 3.44484974781P - 954.239365696$$

To get an i.r.r. of $i = 0.14$, we need this to equal 0. That is, we need to solve

$$3.44484974781P - 954.239365696 = 0$$
$$P = \frac{954.239365696}{3.44484974781}$$
$$= 277.00$$

4. *For a 5-year term insurance policy sold to a life aged 52, with the following lifetable:*

x	l_x	d_x
52	*10000.00*	*30.46*
53	*9969.54*	*33.90*
54	*9935.64*	*37.72*
55	*9897.92*	*41.96*
56	*9855.95*	*46.67*

an actuary performs the following profit test without reserves:

Year	*Premium*	*Expenses*	*Interest*	*Expected Death Benefits*	Pr_t
0		*200*			−200
1	*1700*	*0*	*102.00*	1370.70	431.30
2	*1700*	*54*	*99.24*	1530.16	215.08
3	*1700*	*54*	*99.24*	1708.40	36.84
4	*1700*	*54*	*99.24*	1907.67	−162.43
5	*1700*	*54*	*99.24*	2130.84	−385.60

Calculate the reserves needed to ensure that all cash flows are non-negative.

From the interest in the profit test, we see that the interest rate is $i = \frac{102}{1700} = 0.06$.

For Year 5, the reserve needed is $385.60(1.06)^{-1} = 363.773584906$. The expected reserve payment in Year 4 is therefore $\frac{9855.95}{9897.92} \times 363.773584906 = 362.231081293$. The expected loss in Year 4 is therefore $162.43 + 362.231081293 = 524.661081293$. The reserve in Year 4 is $524.661081293(1.06)^{-1} = 494.963284239$. The expected reserve payment in Year 3 is therefore $\frac{9897.92}{9935.64} \times 494.963284239 = 493.084188873$. The loss in Year 3 is $493.084188873 - 36.84 = 456.244188873$. The reserve for Year 3 is therefore $456.244188873(1.06)^{-1} = 430.419046107$. The expected reserve payment in Year 2 is therefore $\frac{9935.64}{9969.54} \times 430.419046107 = 428.95546748$. Therefore, the expected loss is $428.95546748 - 215.08 = 213.87546748$. This means the reserve in Year 2 is $213.87546748(1.06)^{-1} = 201.769308943$ and the expected reserve payment in Year 1 is $0.996954 \times 201.769308943 = 201.154719628$.

The new table with reserves is

Year	Initial Reserves	Premium	Expenses	Interest	Expected Death Benefits	Expected Reserve Payments	Pr_t
0		200			−200		
1	0	1700	0	102.00	1370.70	201.15	230.15
2	201.77	1700	54	110.87	1530.16	428.96	0
3	430.42	1700	54	124.59	1708.40	493.08	0
4	494.96	1700	54	128.46	1907.67	362.23	0
5	363.77	1700	54	120.59	2130.84	0	0

Standard Questions

5. *An insurer sells a 5-year disability income protection policy for a life aged 48. The transition probabilities are given in the following table:*

x	p_x^{01}	p_x^{02}	p_x^{10}	p_x^{12}
48	*0.002032*	*0.001362*	*0.13453*	*0.06150*
49	*0.002143*	*0.001481*	*0.13264*	*0.07404*
50	*0.002305*	*0.001635*	*0.13007*	*0.08787*
51	*0.002521*	*0.001810*	*0.12635*	*0.10318*
52	*0.002730*	*0.002024*	*0.12395*	*0.12466*

The probability of being in each state at the end of each year is

t	${}_tp_{48}^{00}$	${}_tp_{48}^{01}$
1	0.996606	*0.001362*
2	0.993174955536	0.002556475326
3	0.989594366957	0.00362315814575
4	0.985766219786	0.00458270046075
5	0.981647912899	0.00543858612805

The policy pays a benefit of \$35,000 at the end of any year if the life is disabled at that time (State 1), and pays a death benefit of \$540,000 at the end of the year if the life is dead (State 2). The interest rate is $i = 0.04$. Initial expenses are \$600 plus 30% of the first premium. Renewal expenses are 2.5% of each subsequent premium. The premium is \$1,960 at the start of each year. Use a profit test to calculate the reserves for each year in each state using a reserve rate of $i = 0.05$ and calculate the profit margin at a risk discount rate of $i = 0.12$.

First we perform a profit test without reserves at the reserve rate $i = 0.05$ in the healthy state:

Year	Premium	Expenses	Interest	Expected Death Benefits	Expected Dis Benefits	Pr_t
0		1188			−1188	
1	1960	0	98	735.48	71.12	1251.4
2	1960	49	95.55	799.74	75.005	1131.805
3	1960	49	95.55	882.90	80.675	1042.975
4	1960	49	95.55	977.40	88.235	940.915
5	1960	49	95.55	1092.96	95.550	818.04

and in the disabled state:

Year	Premium	Expenses	Interest	Expected Death Benefits	Expected Dis Benefits	Pr_t
2	0	0	0	39981.60	27766.20	−67747.80
3	0	0	0	47449.80	27372.10	−74821.90
4	0	0	0	55717.20	26966.45	−82683.65
5	0	0	0	67316.40	26298.65	−93615.05

Now we compute the reserve payments in both states:

	Healthy			Disabled		
Year	Loss without Reserves	Reserve	Expected Reserve Payment	Loss without Reserves	Reserve	Expected Reserve Payment
1	−1251.4	0	271.706061121			
2	−1131.805	0	264.563114963	67747.80	199490.500089	141717.225093
3	−1042.975	0	235.714976707	74821.90	178638.160002	112748.168002
4	−940.915	0	161.374514762	82683.65	144168.181472	68692.9405462
5	−818.04	0	0	93615.05	89157.190476	0

With these reserves, we adjust the profit tests, also using the interest rate $i = 0.04$. In the healthy state:

Year	Premium	Expenses	Interest	Expected Death Benefits	Expected Dis Benefits	Expected Res. Payments	Pr_t
0		1188			−1188		
1	1960	0	78.40	735.48	71.12	271.706061121	960.093938879
2	1960	49	76.44	799.74	75.005	264.563114963	848.131885037
3	1960	49	76.44	882.90	80.675	235.714976707	788.150023293
4	1960	49	76.44	977.40	88.235	161.374514762	760.430485238
5	1960	49	76.44	1092.96	95.550	0	798.93

and in the disabled state:

Year	Initial Reserve	Premium	Expenses	Interest	Expected Death Benefits	Expected Dis Benefits	Expected Res. Payment	Pr_t
2	199490.50	0	0	7979.62	39981.60	27766.20	141717.23	−1994.91
3	178638.16	0	0	7145.53	47449.80	27372.10	112748.17	−1786.38
4	144168.18	0	0	5766.73	55717.20	26966.45	68692.94	−1441.68
5	89157.19	0	0	3566.29	67316.40	26298.65	0	−891.57

The profit signature is then calculated as follows:

	Healthy		Disabled		
Year	Pr_t	P(in force)	Pr_t	P(in force)	Π_t
0	−1188	1	0	0	−1188
1	960.093938879	1	—	0	960.093938879
2	848.131885037	0.996606	−1994.905	0.001362	842.536264809
3	788.150023293	0.993174955536	−1786.3816	0.002556475326	778.204023857
4	760.430485238	0.989594366957	−1441.6818152	0.00362315814575	747.294283442
5	798.93	0.985766219786	−891.571905	0.00458270046075	783.472398994

At risk discount rate $i = 0.12$, the NPV is

$$960.09(1.12)^{-1} + 842.54(1.12)^{-2} + 778.20(1.12)^{-3} + 747.29(1.12)^{-4} + 783.47(1.12)^{-5} - 1188 = 1814.28404293$$

The EPV of premiums is

$$1960\left(1+0.996606(1.12)^{-1}+0.993175(1.12)^{-2}+0.989594(1.12)^{-3}+0.985766(1.12)^{-4}\right)=7864.35448394$$

Therefore the profit margin is $\frac{1814.28404293}{7864.35448394}=23.0697\%$.