ACSC/STAT 4720, Life Contingencies II
FALL 2021
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Sample Final Examination
Model Solutions

1. *A life aged 38 wants to buy a 3-year term insurance policy. A life-table based on current-year mortality is:*

x	l_x	d_x
38	10000.00	5.00
39	9995.00	5.14
40	9989.86	5.30
41	9984.56	5.47
42	9979.09	5.67
43	9973.42	5.87

The insurance company uses a single-factor scale function $q(x,t) = q(x,0)(1-\phi_x)^t$ to model changes in mortality. The insurance company uses the following values for ϕ_x:

x	ϕ_x
38	0.03
39	0.025
40	0.025
41	0.02
42	0.015
43	0.02

Calculate $A^{1}_{38:\overline{3}|}$ at interest rate $i = 0.06$, taking into account the change in mortality.

We have $q_{38} = 0.0005$, $q_{39}(1-\phi_{39}) = 0.975 \times \frac{5.14}{9995.00} = 0.00050140070035$, $q_{40}(1-\phi_{40})^2 = 0.975^2 \times \frac{5.30}{9989.86} = 0.000504342653451$. This gives $A^{1}_{38:\overline{3}|} = 0.0005(1.06)^{-1} + (1-0.0005) \times 0.00050140070035(1.06)^{-2} + (1-0.0005)(1-0.00050140070035) \times 0.000504342653451(1.06)^{-3} = 0.00134075170342$.

2. *The following lifetable applied in 2016:*

x	l_x	d_x
55	10000.00	10.63
56	9989.37	11.30
57	9978.07	12.02
58	9966.05	12.80
59	9953.25	13.66
60	9939.59	14.60

An insurance company uses the following mortality scale based on both age and year:

				t		
x	*2017*	*2018*	*2019*	*2020*	*2021*	*2022*
55	0.01	0.015	0.015	0.02	0.02	0.015
56	0.03	0.03	0.025	0.02	0.015	0.02
57	0.02	0.03	0.03	0.025	0.02	0.015
58	0.025	0.03	0.025	0.015	0.015	0.02
59	0.015	0.02	0.015	0.01	0.015	0.01
60	0.02	0.015	0.01	0.015	0.02	0.025

Use this mortality scale to calculate $A^1_{55:\overline{4}|}$ *at interest rate* $i = 0.03$.

For this individual we have

$$q_{55} = (1-0.01)0.001063 = 0.00105237$$
$$q_{56} = (1-0.03)(1-0.03) \times \frac{11.30}{9989.37} = 0.00106434840234$$
$$q_{57} = (1-0.02)(1-0.03)(1-0.03) \times \frac{12.02}{9978.07} = 0.00111077850125$$
$$q_{58} = (1-0.025)(1-0.03)(1-0.025)(1-0.015) \times \frac{12.80}{9966.05} = 0.00116655200405$$

This gives

$$A^1_{55:\overline{4}|} = 0.00105237(1.03)^{-1} + (1-0.00105237)\big(0.00106434840234(1.03)^{-2}$$
$$+(1-0.00106434840234)\big(0.00111077850125(1.03)^{-3} + (1-0.00111077850125) \times 0.00116655200405(1.03)^{-4}\big)\big)$$
$$= 0.00407140699034$$

3. *A pensions company has the current mortality scale for 2017:*

x	$\phi(x, 2017)$	$\left.\frac{d\phi(x,t)}{dt}\right\vert_{x,t=2017}$	$\left.\frac{d\phi(x+t,t)}{dt}\right\vert_{x,t=2017}$
51	*0.016389776*	0.00054272913	−0.0015000971
52	*0.018738397*	−0.00107674028	0.0012410504
53	*0.028229446*	0.00120650853	−0.0002976607
54	*0.028011768*	−0.00109930339	−0.0004183465
55	*0.014334489*	−0.00194027424	0.0023952205
56	*0.016770205*	0.00271342277	−0.0053102487

Mortality in 2016 is given in the following lifetable.

x	l_x	d_x
51	*10000.00*	*15.29*
52	*9984.71*	*16.44*
53	*9968.27*	*17.70*
54	*9950.56*	*19.09*
55	*9931.48*	*20.60*
56	*9910.88*	*22.26*

The company assumes that from 2030 onwards, we will have $\phi(x,t) = 0.01$ for all x and t. Calculate $q(54, 2018)$ using the average of age-based and cohort-based effects.

We fit a cubic curve between the known points. For age 54, we have $\phi(54, 2017+t) = f(t) = at^3 + bt^2 + ct + d$, and we get

$$\begin{aligned} f(0) &= 0.028011768 \\ f'(0) &= -0.00109930339 \\ f(13) &= 0.01 \\ f'(13) &= 0 \end{aligned}$$

We solve this to get

$$\begin{aligned} d &= 0.028011768 \\ c &= -0.00109930339 \\ 13^3a + 13^2b + 13c + d &= 0.01 \\ 3 \times 13^2a + 2 \times 13b + c &= 0 \\ 13^3a - 13c - 2d &= -0.02 \\ a &= \frac{13 \times -0.00109930339 + 2 \times 0.028011768 - 0.02}{13^3} = 0.00000989193988621 \\ b &= \frac{0.00109930339 - 3 \times 13^2 \times 0.00000989193988621}{2 \times 13} = -0.000150611928166 \end{aligned}$$

This gives $f(1) = 0.00000989193988621 - 0.000150611928166 - 0.00109930339 + 0.028011768 = 0.0267717446217$.

For the cohort-based curve, we have $\phi(53+t, 2017+t) = g(t) = \tilde{a}t^3 + \tilde{b}t^2 + \tilde{c}t + \tilde{d}$ and we get

$$\begin{aligned} g(0) &= 0.028229446 \\ g'(0) &= -0.0002976607 \\ g(13) &= 0.01 \\ g'(13) &= 0 \end{aligned}$$

We solve this to get

$$\begin{aligned}
\tilde{d} &= 0.028229446 \\
\tilde{c} &= -0.0002976607 \\
13^3\tilde{a} + 13^2\tilde{b} + 13\tilde{c} + \tilde{d} &= 0.01 \\
3 \times 13^2\tilde{a} + 2 \times 13\tilde{b} + \tilde{c} &= 0 \\
13^3\tilde{a} - 13\tilde{c} - 2\tilde{d} &= -0.02 \\
\tilde{a} &= \frac{13 \times -0.0002976607 + 2 \times 0.028229446 - 0.02}{13^3} = 0.0000148335470642 \\
\tilde{b} &= \frac{0.0002976607 - 3 \times 13^2 \times 0.0000148335470642}{2 \times 13} = -0.00027780567929
\end{aligned}$$

This gives $g(1) = 0.0000148335470642 - 0.00027780567929 - 0.0002976607 + 0.028229446 = 0.0276688131678$. Taking the average of the age-based and cohort-based improvement factors, we get

$$\phi(54, 2018) = \frac{0.0267717446217 + 0.0276688131678}{2} = 0.0272202788948$$

We therefore have

$$\begin{aligned}
q(54, 2018) &= q(54, 2016)(1 - \phi(54, 2017))(1 - \phi(54, 2018)) \\
&= \frac{19.09}{9950.56}(1 - 0.028011768)(1 - 0.0272202788948) \\
&= 0.00181398595891
\end{aligned}$$

4. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.6 \qquad \sigma_k = 1.4 \qquad K_{2017} = -4.83$$

And the following values of α_x and β_x:

x	α_x	β_x
34	−5.314675	0.2697754
35	−5.234098	0.2504377
36	−5.043921	0.1782635
37	−4.892803	0.2889967
38	−4.637988	0.1460634
39	−4.413315	0.1174245
40	−4.261060	0.2078267

The insurance company simulates the following values of Z_t:

t	Z_t
2018	0.2525295
2019	−0.6276655
2020	−0.6007807

Using these simulated values, calculate the probability that a life aged exactly 36 at the start of 2017 dies within the next 4 years.

From the simulated values we have

$$K_{2018} = -4.83 - 0.6 + 1.4 \times 0.2525295 = -5.0764587$$
$$K_{2019} = -5.0764587 - 0.6 + 1.4 \times -0.6276655 = -6.5551904$$
$$K_{2020} = -6.5551904 - 0.6 + 1.4 \times -0.6007807 = -7.99628338$$

This gives us

$$\log(m(36, 2017)) = -5.043921 + 0.1782635 \times -4.83 = -5.904933705$$
$$\log(m(37, 2018)) = -4.892803 + 0.2889967 \times -5.0764587 = -6.35988281199$$
$$\log(m(38, 2019)) = -4.637988 + 0.1460634 \times -6.5551904 = -5.59546139747$$
$$\log(m(39, 2020)) = -4.413315 + 0.1174245 \times -7.99628338 = -5.35227457776$$

Under UDD, we have $m_x = \frac{2q_x}{2-q_x}$ so $q_x = \frac{2m_x}{2+m_x}$. This gives us

$$q(36, 2017) = \frac{2e^{-5.904933705}}{2 + e^{-5.904933705}} = 0.00272225211385$$
$$q(37, 2018) = \frac{2e^{-6.35988281199}}{2 + e^{-6.35988281199}} = 0.001728074972$$
$$q(38, 2019) = \frac{2e^{-5.59546139747}}{2 + e^{-5.59546139747}} = 0.00370779834235$$
$$q(39, 2020) = \frac{2e^{-5.35227457776}}{2 + e^{-5.35227457776}} = 0.00472616844589$$

The probability that the life survives four years is therefore

$$(1-0.00272225211385)(1-0.001728074972)(1-0.00370779834235)(1-0.00472616844589) = 0.987175350394$$

5. *An insurance company uses a Lee-Carter model. One actuary fits the following parameters:*

$$c = -0.13 \qquad \sigma_k = 0.9 \qquad K_{2017} = -1.70 \qquad \alpha_{52} = -4.45 \qquad \beta_{52} = 0.49$$

A second actuary fits the parameters

$$c = -0.14 \qquad \sigma_k = 0.8 \qquad K_{2017} = -1.40 \qquad \alpha_{52} = -4.94 \qquad \beta_{52} = 0.37$$

The insurance company sets its life insurance premiums for 2025 so that under the first actuary's model, it has a 95% chance of an expected profit. What is the probability that these premiums lead to an expected profit under the second actuary's model?

Since expected profit is a decreasing function of $m(x,t)$, we need to calculate the probability under the second actuary's model that $\log(m(52,2025))$ is less than the 95th percentile of the first actuary's distribution for $\log(m(52,2025))$.

The first actuary's model gives $\log(m(52,2025)) = -4.45 + 0.49K_{2025}$, where $K_{2025} = -1.70 - 0.13 \times 8 + 0.9(Z_{2018} + Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023} + Z_{2024} + Z_{2025}) \sim N(-2.74, 6.48)$. The 95th percentile of this model is therefore $\log(m(52,2025)) = -4.45 + 0.49(-2.74 + 1.644854\sqrt{6.48}) = -3.7409137956$. The second actuary's model gives $\log(m(52,2025)) = -4.94 + 0.37K_{2025}$, where $K_{2025} = -1.40 - 0.14 \times 8 + 0.8(Z_{2018} + Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023} + Z_{2024} + Z_{2025}) \sim N(-2.52, 5.12)$. Under this model $\log(m(52,2025)) \sim N(-5.8724, 5.12 \times 0.37^2)$. The probability that $\log(m(52,2025)) < -3.7409137956$ is therefore

$$\Phi\left(\frac{-3.7409137956 - (-5.8724)}{0.37\sqrt{5.12}}\right) = \Phi(2.54592626543) = 0.9945506$$

6. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$$K^{(1)}_{2017} = -3.29 \qquad K^{(2)}_{2017} = 0.38 \qquad c^{(1)} = -0.17 \qquad c^{(2)} = 0.01$$
$$\sigma_{k_1} = 0.5 \qquad \sigma_{k_2} = 0.08 \qquad \rho = 0.3 \qquad \overline{x} = 47$$

What is the probability that the mortality for an individual currently (in 2017) aged 39 will be higher in 2025 than in 2030?

The mortality in 2025 satisfies

$$\log\left(\frac{q(47,2025)}{1-q(47,2025)}\right) = K^{(1)}_{2025}$$

while mortality in 2030 satisfies

$$\log\left(\frac{q(52,2030)}{1-q(52,2030)}\right) = K^{(1)}_{2030} + 5K^{(2)}_{2030}$$

Since $\log\left(\frac{q}{1-q}\right)$ is an increasing function of q, we are asking, what is the probability that $K^{(1)}_{2025} > K^{(1)}_{2030} + 5K^{(2)}_{2030}$. We have that $K^{(1)}_{2030} = K^{(1)}_{2025} - 5 \times 0.17 + 0.5(Z^{(1)}_{2026} + Z^{(1)}_{2027} + Z^{(1)}_{2028} + Z^{(1)}_{2029} + Z^{(1)}_{2030})$ and $K^{(2)}_{2030} = K^{(2)}_{2017} + 13 \times 0.01 + 0.08(Z^{(2)}_{2018} + \cdots + Z^{(2)}_{2030})$. The probability that we are interested in is therefore the probability that

$$-0.85 + 0.5(Z^{(1)}_{2026} + Z^{(1)}_{2027} + Z^{(1)}_{2028} + Z^{(1)}_{2029} + Z^{(1)}_{2030}) + 5\left(0.51 + 0.08(Z^{(2)}_{2018} + \cdots + Z^{(2)}_{2030})\right) < 0$$
$$1.7 + 0.5(Z^{(1)}_{2026} + Z^{(1)}_{2027} + Z^{(1)}_{2028} + Z^{(1)}_{2029} + Z^{(1)}_{2030}) + 0.4(Z^{(2)}_{2018} + \cdots + Z^{(2)}_{2030}) < 0$$

We have that $\mathrm{Cov}(Z_t^{(1)}, Z_t^{(2)}) = 0.3$, so $\mathrm{Var}(0.5Z_t^{(1)} + 0.4Z_t^{(2)}) = 0.5^2 + 0.4^2 + 2 \times 0.3 \times 0.4 \times 0.5 = 0.53$. We therefore get that

$$\log\left(\frac{q(52,2030)}{1-q(52,2030)}\right) - \log\left(\frac{q(47,2025)}{1-q(47,2025)}\right)) \sim N(1.7, 3.91)$$

The probability that it is less than 0 is therefore $\Phi\left(\frac{-1.7}{\sqrt{3.91}}\right) = \Phi(-0.85972695362) = 0.1949698$.

7. *An individual aged 42 has a current salary of \$76,000 for the coming year. The salary scale is $s_y = 1.05^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

   If the individual retires at age 65, the final average salary is $76000\frac{((1.05)^{20}+(1.05)^{21}+(1.05)^{22})}{3} = \$211,901.20$.

8. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 60% of final average salary for an employee who enters the plan at exact age 30, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 60% of the life annuity.*
   - *At age 65, the employee is married to someone aged 63.*
   - *The salary scale is $s_y = 1.04^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000016(1.092)^x$. The value of the life annuity is based on $\delta = 0.045$. This gives $\overline{a}_{65} = 19.63036$, $\overline{a}_{63} = 19.83656$ and $\overline{a}_{65,63} = 18.7867$.*
   - *A fixed percentage of salary is payable annually in arrear.*
   - *Contributions earn an annual rate of 7%.*

   Calculate the percentage of salary payable annually to achieve the target replacement rate under these assumptions.

   Suppose the employee has a current salary at age 30 for the coming year of 1. At retirement, the employee's final average salary is $\frac{1.04^{32}+1.04^{33}+1.04^{34}}{3} = 3.650252$. With a replacement ratio of 60%, this leads to an annuity of $0.6 \times 3.650252 = 2.1901512$ and a reversionary annuity of $0.6 \times 2.1901512 = 1.31409072$. For the reversionary annuity, $\overline{a}_{65|63} = \overline{a}_{63} - \overline{a}_{65,63} = 19.83656 - 18.7867 = 1.04986$, so the total EPV of the annuities at age 65 is $2.1901512 \times 19.63036 + 1.31409072 \times 1.04986 = 44.37307$.

   The contributions should therefore be chosen so that the accumulated value of contributions by age 65 is 44.37307. If the employee pays all salary into the pension plan, the accumulated value will be $\frac{1.07^{35}-1.04^{35}}{1.07-1.04} = 224.3497$. The percentage of salary that needs to be paid into the plan is therefore $\frac{44.37307}{224.3497} = 19.78\%$.

9. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	1.000000	39	1.350398	48	1.845766	57	2.553877
31	1.033333	40	1.397268	49	1.912422	58	2.649694
32	1.067933	41	1.445983	50	1.981785	59	2.749515
33	1.103853	42	1.496620	51	2.053975	60	2.853522
34	1.141149	43	1.549263	52	2.129115	61	2.961903
35	1.179879	44	1.604000	53	2.207337	62	3.074855
36	1.220103	45	1.660921	54	2.288777	63	3.192585
37	1.261887	46	1.720122	55	2.373580	64	3.315310
38	1.305295	47	1.781702	56	2.461894	65	3.443256

An employee aged 42 and 4 months has 12 years of service, and a current salary of \$106,000 (for the coming year). She has a defined benefit pension plan with $\alpha = 0.02$ and S_{Fin} is the average of her last 3 years' salary. The employee's mortality is given by $\mu_x = 0.00000195(1.102)^x$. The pension benefit is payable monthly in advance. The interest rate is $i = 0.05$. This results in $\ddot{a}_{65}^{(12)} = 17.15373$ and ${}_{22.66666667}p_{42.33333333} = 0.9901951$. There is no death benefit, and there are no exits other than death or retirement at age 65.

(a) Calculate the EPV of the accrued benefit using the projected unit method under the assumption that the employee retires at age 65. [Calculate the salary scale at non-integer ages by linear interpolation.]

We interpolate $s_{42.33333} = \frac{2}{3} \times 1.496620 + \frac{1}{3} \times 1.549263 = 1.514168$. The employee's final average salary is therefore $106000 \times \frac{3.192585+3.315310+3.443256}{1.514168\times 3} = 232211.57$. The accrued benefit is a pension with annual payment rate $0.02 \times 12 \times 232211.57\times = \$55,730.78$. The EPV of this benefit when the employee reaches age 65 is therefore $55730.78 \times 17.15373 = 955990.75$. The current EPV is $955990.75 \times 0.9901951(1.05)^{-22.666667} = \$313,244.67$.

(b) Calculate the employer's contribution for this employee for the year. [${}_{21.66666667}p_{43.33333333} = 0.9903189$.]

In a year's time, the accrued benefit will be a pension with annual payment rate $0.02 \times 13 \times 232211.57\times = \$60,375.01$. The EPV of this is $60375.01 \times 17.15373 \times 0.9903189(1.05)^{-21.666666667} = 356360.35$. Discounting to the start of year, and multiplying by the probability of surviving gives $356360.35(1.05)^{-1} \times \frac{0.9901951}{0.9903189} = 339348.38$. The annual contribution is therefore $339348.38 - 313,244.67 = \$26,103.71$.

The accumulated value of the previous contributions is $313244.67\times 1.05 = 328906.90$, so this year's contribution is $356360.35 - 328906.90 = \$27,453.45$.

10. *The service table is given below:*

x	l_x	1	2	3
40	*10000.00*	*118.76*	*0*	*0.51*
41	*9880.73*	*112.29*	*0*	*0.58*
42	*9767.86*	*107.16*	*0*	*0.65*
43	*9660.05*	*101.84*	*0*	*0.73*
44	*9557.49*	*96.80*	*0*	*0.82*
45	*9459.86*	*92.02*	*0*	*0.93*
46	*9366.91*	*87.50*	*0*	*1.04*
47	*9278.37*	*83.19*	*0*	*1.18*
48	*9193.99*	*80.11*	*0*	*1.32*
49	*9112.57*	*75.21*	*0*	*1.49*
50	*9035.87*	*71.48*	*0*	*1.68*
51	*8962.71*	*67.92*	*0*	*1.89*
52	*8892.90*	*64.51*	*0*	*2.12*
53	*8826.26*	*61.23*	*0*	*2.39*
54	*8762.64*	*58.07*	*0*	*2.69*
55	*8701.88*	*55.03*	*0*	*3.03*
56	*8643.83*	*52.06*	*0*	*3.41*
57	*8588.36*	*49.18*	*0*	*3.84*
58	*8535.34*	*46.37*	*0*	*4.32*
59	*8484.64*	*43.62*	*0*	*4.86*
60^-	*8484.64*		*1098.84*	
60	*7385.80*	*21.70*	*819.91*	*5.79*
61	*6538.40*	*18.30*	*611.98*	*6.38*
62	*5901.74*	*10.81*	*384.29*	*5.86*
63	*5500.78*	*9.14*	*639.20*	*6.15*
64	*4846.29*	*7.73*	*351.32*	*6.10*
65^-	*4481.14*		*4481.14*	

The salary scale is $s_y = 1.05^y$*. The accrual rate is 0.02. The benefit for employees who withdraw is a deferred annual pension with COLA 2%, starting from age 65. For an individual aged 65, we have* $\ddot{a}_{65} = 12.85$*. The interest rate is* $i = 0.04$*. The lifetable for an individual who has withdrawn is*

x	l_x	d_x
57	*10000.00*	*7.54*
58	*9992.46*	*8.22*
59	*9984.24*	*8.95*
60	*9975.29*	*9.76*
61	*9965.52*	*10.65*
62	*9954.87*	*11.63*
63	*9943.25*	*12.69*
64	*9930.55*	*13.86*
65	*9916.69*	*15.15*

Calculate the EPV of deferred pension benefits made to an individual aged exactly 57, with 16 years of service, whose salary for the past year was $121,000.

The current final average salary is $121,000\frac{1+1.05^{-1}+1.05^{-2}}{3} = 115329.55$, so if the individual withdraws at age t, then the annual accrued pension benefits are $0.32 \times 115329.55(1.05)^t(1.02)^{8-t} = 43240.62\left(\frac{1.05}{1.02}\right)^t$. The EPV of the accrued deferred pension benefits valued at a time 8 years in the future is therefore

$$43240.62\left(\frac{1.05}{1.02}\right)^t {}_{8-t}p_{57+t}\frac{d^{01}_{57+t}}{l_{57}}\ddot{a}_{65}$$

We calculate

$$12.85\times 43240.62\times\frac{1}{8588.36}\times 9916.69(1.04)^{-8}=468796.998406$$

Therefore, the current EPV of withdrawl benefits due to withdrawing in t years is $468796.998406\frac{d^{01}_t}{l_t}$, where d^{01}_t is from the service table and l_t is from the lifetable. We calculate the following table

t	d^{01}_t	l_t	$\left(\frac{1.05}{1.02}\right)^t$	EPV of accrued withdrawl benefit
0.5	49.18	9996.23	1.01459931239	2273.226
1.5	46.37	9988.35	1.04444046863	2083.745
2.5	43.62	9979.765	1.07515930594	1905.800
3.5	21.70	9970.405	1.10678163847	921.870
4.5	18.30	9960.195	1.13933403959	755.992
5.5	10.81	9949.06	1.17284386428	434.298
6.5	9.14	9936.90	1.20733927205	357.150
7.5	7.73	9923.62	1.24284925064	293.816

The total EPV of accrued deferred pension benefits is therefore

$$2273.226+2083.745+1905.800+921.870+755.992+434.298+357.150+293.816=\$9,025.90$$

11. *An insurance company sells a 5-year annual life insurance policy to a life aged 53, for whom the lifetable below is appropriate.*

x	l_x	d_x
53	*10000.00*	*49.24*
54	*9950.76*	*54.62*
55	*9896.14*	*60.60*
56	*9835.55*	*67.22*
57	*9768.32*	*74.56*
58	*9693.76*	*82.68*

The annual gross premium is \$685. Initial expenses are \$400. The death benefits are \$90,000. Renewal costs are 2% of each subsequent premium. The interest rate is $i = 0.05$

(a) Calculate the profit vector for the policy.

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		400			−400
1	685	0.00	34.250	443.16	276.09
2	685	13.70	33.565	494.01	210.85
3	685	13.70	33.565	551.12	153.74
4	685	13.70	33.565	615.10	89.77
5	685	13.70	33.565	686.96	17.91

The profit vector is the last column of this table.

(b) Calculate the discounted payback period of the policy using a risk discount rate $i = 0.07$.

Using a risk discount rate of $i = 0.07$, we get the following partial NPVs:

t	P(in force)	Discounted Pr_t	NPV(t)
0	1	−400.00	−400.00
1	1.000000	258.02	−141.98
2	0.995076	183.26	41.28
3	0.989614	124.20	164.19
4	0.983554	67.36	230.44
5	0.976832	12.47	242.62

So the discounted payback period is 2 years.

12. *An insurance company sells a 5-year endowment insurance policy to a life aged 35 for whom the lifetable below is appropriate.*

x	l_x	d_x
35	*10000.00*	*8.74*
36	*9991.26*	*9.45*
37	*9981.81*	*10.24*
38	*9971.57*	*11.12*
39	*9960.45*	*12.11*
40	*9948.35*	*13.22*

The benefit is \$300,000. The annual premium is \$60,000, and the interest rate is $i = 0.03$. Initial expenses are \$2,400 and renewal expenses are \$80 at the start of each year after the first. Use a profit test to calculate the reserves at the start of each year. There are no exits other than death or maturity.

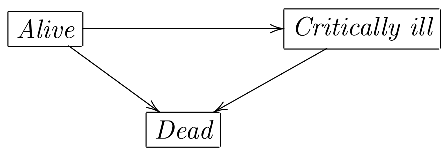
We first perform a profit test without reserves.

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Expected Maturity Benefit	Net Cash Flow
0		2400				−2400
1	60000	0	1800.00	262.20	0.00	61537.80
2	60000	80	1797.60	283.75	0.00	61433.85
3	60000	80	1797.60	307.76	0.00	61409.84
4	60000	80	1797.60	334.55	0.00	61383.05
5	60000	80	1797.60	364.74	299635.26	−238282.40

The reserves at the begining of the fifth year are therefore $238282.40(1.03)^{-1} = 231342.14$. The expected reserve payments at the end of the fourth year are $231342.14 \times \frac{9960.45}{9971.57} = 231084.15$. This makes the net cash flow at end of fourth year with no reserve $61383.05 - 231084.15 = -169701.10$, so the reserve for the fourth year is $169701.10(1.03)^{-1} = 164758.35$. The expected reserve payment at the end of the third year is $164758.35 \times \frac{9971.57}{9981.81} = 164589.33$. This makes a net cash flow of $61409.84 - 164589.33 = -103179.49$ at the end of the third year, so the reserve for the third year is $103179.49(1.03)^{-1} = 100174.26$. The expected reserve payment at the end of the second year is therefore $100174.26 \times \frac{9981.81}{9991.26} = 100079.51$, and the expected cash-flow at the end of the second year is $61433.85 - 100079.51 = -38645.66$. This means that the reserve at the beginning of the second year is $38645.66(1.03)^{-1} = 37520.06$. The expected reserve payment at the end of the first year is therefore $37520.06 \times 0.999126 = 37487.27$. We therefore have the following table:

t	Premium (at $t-1$)	Reserve	Expenses	Interest	Exp. Death Benefits	Exp. Mat. Benefit	Exp. Res. Payment	Net Cash Flow
0		2400						-2400
1	60000	0.00	0	1800.00	262.20	0.00	37487.27	24050.53
2	60000	37520.06	80	1797.60	283.75	0.00	100079.51	0.00
3	60000	100174.26	80	1797.60	307.76	0.00	164589.33	0.00
4	60000	164758.35	80	1797.60	334.55	0.00	231084.15	0.00
5	60000	231342.14	80	1797.60	364.74	299635.26	0.00	0.00

13. *An insurance company offers a 5-year critical illness insurance policy. The policy has 3 states — alive, critically ill, and dead. The possible transitions are as shown in the following diagram:*



Premiums are payable at the start of each year while in the alive state.

For a life aged 37, transitions are as shown in the following lifetable:

age	*Alive*	*Critically Ill*	*Death (direct)*	*Death (critically ill)*	*CI and Death*
37	*10000.00*	*0.00*	*6.95*	*0.00*	*0.03*
38	*9990.20*	*2.82*	*7.47*	*0.03*	*0.04*
39	*9979.47*	*6.01*	*8.03*	*0.08*	*0.03*
40	*9967.71*	*9.63*	*8.66*	*0.15*	*0.04*
41	*9954.78*	*13.75*	*9.36*	*0.22*	*0.03*

At the end of 5 years, the expected number of lives who are critically ill is 18.42.

Initial expenses are 28% of the first premium, and renewal expenses are 4% of subsequent premiums while the life is in the alive state. There are also renewal expenses of \$80 at the start of each year if the life is in the critically ill state. Premiums are payable at the start of each year when the life is in the healthy state. There is a death benfit of \$250,000 at the end of the year in which the life dies, and a benefit of \$100,000 at the end of the year in which the life becomes critically ill. (If the life becomes critically ill and then dies later in the same year, both benefits are payable at the end of the year.) The interest rate is $i = 0.04$. *Use a profit test without reserves to determine the premium for this policy which achieves a profit margin of 5% at a risk discount rate of* $i = 0.10$.

We first perform a profit test for lives which start the year in the critically ill state, and in the alive state alive state, with the premium set as P.

Critically Ill:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
2	0	80	-3.20	2659.57	-2742.77
3	0	80	-3.20	3327.79	-3410.99
4	0	80	-3.20	3894.08	-3977.28
5	0	80	-3.20	4000.00	-4083.20

Alive:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Expected CI Benefits	Net Cash Flow
0		$0.28P$			$-0.28P$	
1	P	0	$0.04P$	174.50	28.50	$1.04P-203.00$
2	P	$0.04P$	$0.0384P$	187.93	32.63	$0.9984P-220.57$
3	P	$0.04P$	$0.0384P$	201.91	37.38	$0.9984P-239.29$
4	P	$0.04P$	$0.0384P$	218.20	43.24	$0.9984P-261.44$
5	P	$0.04P$	$0.0384P$	235.82	49.42	$0.9984P-285.24$

The profit vector is then calculated in the usual way

t	Pr_t (Alive)	${}_{t-1}p^{00}$	Pr_t (Alive)	${}_{t-1}p^{00}$	Π_t
0	$-0.28P$	1		0	$-0.28P$
1	$1.04P-203.00$	1		0	$1.04P-203.00$
2	$0.9984P-220.57$	0.999020	-2742.77	0.000282	$0.997421568P-221.12730254$
3	$0.9984P-239.29$	0.997947	-3410.99	0.000601	$0.9963502848P-240.84874262$
4	$0.9984P-261.44$	0.996771	-3977.28	0.000963	$0.9951761664P-264.42593088$
5	$0.9984P-285.24$	0.995478	-4083.20	0.001375	$0.9938852352P-289.56454472$

At a risk discount rate of $i=0.10$, the NPV of the policy is

$$\begin{aligned}
&-0.28P+(1.04P-203.00)(1.1)^{-1}+0.999020(0.9984P-220.57)(1.1)^{-2}-0.000282\times 2742.77(1.1)^{-2}\\
&+0.997947(0.9984P-239.29)(1.1)^{-3}-0.000601\times 3410.99(1.1)^{-3}\\
&+0.996771(0.9984P-261.44)(1.1)^{-4}-0.000963\times 3977.28(1.1)^{-4}\\
&+0.995478(0.9984P-285.24)(1.1)^{-5}-0.001375\times 4083.20(1.1)^{-5}\\
&=3.535186P-908.6518
\end{aligned}$$

The NPV of premium payments is $P(1+0.999020(1.1)^{-1}+0.997947(1.1)^{-2}+0.996771(1.1)^{-3}+0.995478(1.1)^{-4})=4.161763P$

The profit margin is therefore $\frac{3.535186P-937.607908.6518}{4.161763P}$. Setting this equal to 0.05 gives

$$\begin{aligned}
\frac{3.535186P-908.6518}{4.161763P}&=0.05\\
3.535186P-908.6518&=0.20808815P\\
3.327098P&=908.6518\\
P&=\$273.11
\end{aligned}$$

14. *A couple purchase a 5-year last survivor insurance policy. Annual Premiums of \$49,830 are payable while both are alive. If one life is dead, there are no premiums or benefits. If both lives die within the 5-year period, a benefit of \$1,000,000 is payable. The husband is 74 and the wife is 81. Their lifetables are given below. Assume both lives are independent.*

x	l_x	d_x
74	10000.00	591.85
75	9408.15	628.62
76	8779.53	662.27
77	8117.26	691.27
78	7425.99	713.96
79	6712.03	728.54

x	l_x	d_x
81	10000.00	1113.81
82	8886.19	1114.43
83	7771.76	1097.45
84	6674.31	1061.21
85	5613.10	1004.92
86	4608.18	928.94

Initial expenses are \$3,000, and renewal expenses are \$80 at the start of each subsequent year while both are alive, and \$60 at the start of each year while only one is alive. The interest rate is $i = 0.04$.

An actuary computes the following profit tests without reserves in each state.

both alive:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		3000.00			−3000.00
1	49830	0	1993.20	6592.08	45231.12
2	49830	80	1990.00	8379.56	43360.44
3	49830	80	1990.00	10651.95	41088.05
4	49830	80	1990.00	13540.45	38199.55
5	49830	80	1990.00	17212.67	34527.33

husband alive wife dead:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		3000.00			−3000.00
1	0	0	0	59185.00	−59185.00
2	0	60	−2.40	66816.54	−66878.94
3	0	60	−2.40	75433.42	−75495.82
4	0	60	−2.40	85160.51	−85222.91
5	0	60	−2.40	96143.41	−96205.81

wife alive husband dead:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		3000.00			−3000.00
1	0	0	0	111381.00	−111381.00
2	0	60	−2.40	125411.45	−125473.85
3	0	60	−2.40	141209.97	−141272.37
4	0	60	−2.40	158999.21	−159061.61
5	0	60	−2.40	179031.19	−179093.59

a) Calculate the reserves in the husband alive, wife dead state and in the husband dead, wife alive state.

We first calculate reserves in the husband alive, wife dead state: In Year 5, the reserves are $96205.81(1.04)^{-1} = 92505.5865384$. The expected reserve payments at the end of Year 4 are $92505.5865384 \times \frac{7425.99}{8117.26} = 84627.7636269$. The net cash flow in Year 4 is therefore $-85222.91 - 84627.7636269 = -169850.673627$ so the reserve in Year 4 is $169850.673627(1.04)^{-1} = 163317.95541$. The expected reserve payment at the end of Year 3 is $163317.95541\frac{8117.26}{8779.53} = 150998.323$. The net cash flow in Year 3 is therefore $-75495.82 - 150998.323 =$

-226494.143. The reserves in Year 3 are $226494.143(1.04)^{-1} = 217782.829808$. The expected reserve payments in Year 2 are therefore $217782.829808\frac{8779.53}{9408.15} = 203231.335362$, so NCF at the end of Year 2 is $-66878.94 - 203231.335362 = -270110.275362$. Therefore Year 2 reserves are $270110.275362(1.04)^{-1} = 259721.418617$.

We next calculate reserves in the husband dead, wife alive state: In Year 5, the reserves are $179093.59(1.04)^{-1} = 172205.375$. The expected reserve payments at the end of Year 4 are $172205.375 \times \frac{5613.10}{6674.31} = 144824.856863$. The net cash flow in Year 4 is therefore $-159061.61 - 144824.856863 = -303886.466863$ so the reserve in Year 4 is $303886.466863(1.04)^{-1} = 292198.52583$. The expected reserve payment at the end of Year 3 is $292198.52583\frac{6674.31}{7771.76} = 250937.18063$. The net cash flow in Year 3 is therefore $-141272.37 - 250937.18063 = -392209.55063$. The reserves in Year 3 are $392209.55063(1.04)^{-1} = 377124.567913$. The expected reserve payments in Year 2 are therefore $377124.567913\frac{7771.76}{8886.19} = 329828.827869$, so NCF at the end of Year 2 is $-125473.85 - 329828.827869 = -455302.677869$. Therefore Year 2 reserves are $455302.677869(1.04)^{-1} = 437791.036412$.

b) Perform a new profit test with these reserves and use it to calculate the reserves in the both alive state.

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Exp. HAWD Reserves	Exp. HDWA Reserves	Net Cash Flow
0		3000.00					−3000.00
1	49830	0	1993.20	6592.08	27215.9257929	23024.7069912	−5009.5127841
2	49830	80	1990.00	8379.56	25487.5370734	22038.0199907	−4165.1170641
3	49830	80	1990.00	10651.95	21322.4687299	18929.0504863	836.5307838
4	49830	80	1990.00	13540.45	13455.7473415	10372.3643967	14371.4382618
5	49830	80	1990.00	17212.67	0	0	34527.33

In Year 2, the reserves are $4165.1170641(1.04)^{-1} = \$4,004.92025394$. The expected reserve payments for the both alive state at the end of Year 1 is $4004.92025394 \times 0.9408 \times 0.888619 = \$3,348.16441585$. With these additional reserve payments, the net cash flow in Year 1 is $-5009.5127841 - 3348.16441585 = -8357.67719995$, so the reserves are $8357.67719995(1.04)^{-1} = \$8,036.22807687$. With these reserves we calculate the following profit test

t	Reserves	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Exp. HAWA Reserves	Exp. HAWD Reserves	Exp. HDWA Reserves	Net Cash Flow
0			3000.00			8036.23			−11036.23
1	8036.23	49830	0.00	1993.20	6592.08	3348.16	27215.93	23024.71	0
2	4,004.92	49830	80	1990.00	8379.56	0.00	25487.54	22038.02	0
3		49830	80	1990.00	10651.95	0.00	21322.47	18929.05	836.53
4		49830	80	1990.00	13540.45	0.00	13455.75	10372.36	14371.44
5		49830	80	1990.00	17212.67	0.00	0.00	0.00	34527.33

c) Calculate the profit signature of the policy.

Since the net cash flows are zero in the husband dead wife alive state, and in the husband alive, wife dead state, we only need to consider cash-flows in the both alive state

t	$\Pr_t^{(0)}$	${}_{t-1}p^{00}_{74,81}$	Π_t
0	-11036.2280769	1	-11036.2280769
1	0	1	0
2	0	0.836026084485	0
3	836.53	0.682324000728	570.78
4	14371.44	0.541771095906	7786.03
5	34527.33	0.41682824469	14391.97

15. *An insurance company collects the following data in a mortality study*

i	d_i	x_i	u_i	i	d_i	x_i	u_i	i	d_i	x_i	u_i
1	*68.0*	*68.1*	-	*8*	*68.8*	*68.9*	-	*15*	*69.4*	-	*69.6*
2	*68.0*	*71.8*	-	*9*	*68.8*	-	*69.0*	*16*	*69.6*	*71.2*	-
3	*68.0*	-	*68.3*	*10*	*69.0*	*69.0*	-	*17*	*70.1*	-	*72.9*
4	*68.0*	-	*69.5*	*11*	*69.1*	*69.2*	-	*18*	*70.5*	*70.5*	-
5	*68.0*	*75.7*	-	*12*	*69.2*	*69.3*	-	*19*	*70.6*	-	*70.6*
6	*68.4*	*68.6*	-	*13*	*69.3*	*69.4*	-	*20*	*70.7*	*70.9*	-
7	*68.6*	*68.7*	-	*14*	*69.4*	*69.5*	-	*21*	*70.7*	-	*71.1*

Using a Nelson-Åalen estimator:

(a) estimate the probability that an individual aged 68 survives to age 71.

We observe the following:

x_i	s_i	r_i	$H(x_i)$
68.1	1	$5-0-0=5$	0.2
68.6	1	$7-1-1=5$	0.4
68.7	1	$7-2-1=4$	0.65
68.9	1	$9-3-1=5$	0.85
69.0	1	$10-4-1=5$	1.05
69.2	1	$12-5-2=5$	1.25
69.3	1	$13-6-2=5$	1.45
69.4	1	$15-7-2=6$	1.6167
69.5	1	$15-8-2=5$	1.8167
70.5	1	$18-9-4=5$	2.0167
70.9	1	$21-10-5=6$	2.1833

This gives $H(71) = 2.1833$, so $S(71) = e^{-2.1833} = 0.11266535285$.

(b) Find a 95% log-transformed confidence interval for ${}_3p_{68}$.

We have that $H(71) = \frac{1}{5}+\frac{1}{5}+\frac{1}{4}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{6}+\frac{1}{5}+\frac{1}{5}+\frac{1}{6} = 2.1833$ and the variance is therefore $\mathrm{Var}(H(71)) = \frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{6^2} = 0.438055555556$. This gives us that $\mathrm{Var}(\log(H(71))) = \frac{0.438055555556}{2.1833^2} = 0.0918972177482$. A 95% confidence interval for $\log(H(71))$ is therefore $\log(2.1833) \pm 1.96 \times \sqrt{0.0918972177482} = [0.186672237465, 1.37500274991]$, which corresponds to the interval $[e^{0.186672237465}, e^{1.37500274991}] = [1.20523219035, 3.95508759904]$ for $H(71)$, and thus, to the interval $[e^{-3.95508759904}, e^{-1.20523219035}] = [0.0191569903422, 0.299622422001]$ for ${}_3p_{68}$.

16. *An insurance company collects the following data in a mortality study*

i	y_i	s_i	r_i	i	y_i	s_i	r_i	i	y_i	s_i	r_i
1	42.1	4	103	7	42.8	4	97	13	43.5	9	91
2	42.2	8	85	8	42.9	8	96	14	43.6	3	70
3	42.3	4	105	9	43.1	6	72	15	43.7	7	105
4	42.5	4	99	10	43.2	4	79	16	43.8	9	98
5	42.6	6	95	11	43.3	8	102	17	43.9	7	88
6	42.7	3	88	12	43.4	4	88				

(a) Use a Kaplan-Meier product-limit estimator to estimate p_{42}.

The Kaplan-Meier estimator is

$$\widehat{p_{42}} = \frac{99}{103} \times \frac{77}{85} \times \frac{101}{105} \times \frac{95}{99} \times \frac{89}{95} \times \frac{85}{88} \times \frac{93}{97} \times \frac{88}{96} = 0.639168793581$$

(b) Find a 95% log-transformed confidence interval for p_{42}, using Greenwood's approximation to estimate the variance.

Using Greenwood's approximation, the estimated variance is

$$\mathrm{Var}(\widehat{p_{42}}) \approx 0.639168793581^2 \left(\frac{4}{99 \times 103} + \frac{8}{77 \times 85} + \frac{4}{101 \times 105} + \frac{4}{95 \times 99} + \frac{6}{89 \times 95} + \frac{3}{85 \times 88} + \frac{4}{93 \times 97} + \frac{8}{88 \times 96} \right) = 0.0020092465977$$

The log-transformed confidence interval is $\left[\hat{p}^{\frac{1}{U}}, \hat{p}^{U}\right]$ where

$$U = e^{\Phi^{-1}\left(\frac{\alpha}{2}\right)\frac{\sigma}{\hat{p}\log(\hat{p})}} = e^{\frac{1.96 \times \sqrt{0.00200924659777}}{0.639168793581 \log(0.639168793581)}} = 0.735576970322$$

This gives the confidence interval

$$\left[0.639168793581^{1.35947703687}, 0.639168793581^{0.735576970322}\right] = [0.544175295383, 0.719474299147]$$

17. *Using the following table:*

Age	*No. at start*	*enter*	*die*	*leave*	*No. at next age*
65	0	46	8	15	23
66	23	38	14	20	27
67	27	53	22	27	31
68	31	44	22	18	35
69	35	38	28	24	21
70	21	32	27	26	0

Assume that events are uniformly distributed over the year. Estimate the probability ${}_2p_{67}$ of a life aged 67 surviving for 2 years using

(a) Exact exposure.

The exact exposure aged 67 is $27 + \frac{53}{2} - \frac{22}{2} - \frac{27}{2} = 29$, while there are 22 deaths, so we estimate $\mu_{67} = \frac{22}{29} = 0.758620689655$. The exact exposure at age 68 is $31 + \frac{44}{2} - \frac{22}{2} - \frac{18}{2} = 33$, and there are 22 deaths, so we estimate $\mu_{68} = \frac{22}{33} = 0.66666667$. We therefore get ${}_2p_{67} = e^{-0.758620689655 - 0.66666667} = 0.240439360707$.

(b) Actuarial exposure.

The actuarial exposure aged 67 is $27 + \frac{53}{2} - \frac{27}{2} = 40$, and there are 22 deaths. The actuarial exposure aged 67 is $31 + \frac{44}{2} - \frac{18}{2} = 44$, and there are 22 deaths. This gives us ${}_2p_{67} = \frac{18}{40} \times \frac{22}{44} = 0.225$.