ACSC/STAT 4720, Life Contingencies II

FALL 2022
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Homework Sheet 6

Model Solutions

Basic Questions

1. *An individual aged 39 has a current salary of \$71,000. The salary scale is $s_y = 1.05^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

   The individual's last 3 years start at ages 62, 63 and 64, which are respectively 23, 24 and 25 years in the future. The final average salary is therefore

   $$71000\left(\frac{1.05^{23}+1.05^{24}+1.05^{25}}{3}\right) = \$229,163.83$$

2. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 65% of final average salary for an employee who enters the plan at exact age 30, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 70% of the life annuity.*
   - *At age 65, the employee is married to someone aged 66.*
   - *The salary scale is $s_y = 1.03^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000016(1.113)^x$.*
   - *A fixed percentage of salary is payable monthly in arrear.*
   - *Contributions earn an annual rate of 5%.*
   - *The value of the life annuity is based on $\delta = 0.045$.*

   Calculate the percentage of salary payable monthly to achieve the target replacement rate under these assumptions. [You may use numerical integration to compute the value of the annuities.]

   Using numerical integration, we get that

   $$\overline{a}_{65} = \int_0^\infty e^{\frac{0.0000016}{\log(1.113)}(1.113^{65}-1.113^{65+t})}e^{-0.045t}\,dt = 16.7706$$

Similarly, we get that

$$\overline{a}_{65|66} = \int_0^\infty \left(1 - e^{\frac{0.0000016}{\log(1.113)}(1.113^{65} - 1.113^{65+t})}\right) e^{\frac{0.0000016}{\log(1.113)}(1.113^{66} - 1.113^{66+t})} e^{-0.045t}\, dt = 1.473165$$

If the initial salary is S, then the employee's projected final average salary is

$$S\left(\frac{1.03^{32} + 1.03^{33} + 1.03^{34}}{3}\right) = 2.65310776318S$$

The expected cost of the annuities at time of retirement is therefore

$$2.65310776318 \times 0.65 \times (16.7706 + 0.7 \times 1.473165)S = 30.6995876861S$$

Making monthly contributions of $\frac{1}{12}$ in arrear, if the first monthly salary is M, then the accumulated value of this individual investing her whole salary in the plan is

$$\frac{1.05^{35} - 1.03^{35}}{1.05^{\frac{1}{12}} - 1.03^{\frac{1}{12}}} M = 1680.59593496M$$

Her initial annual salary is $S = \frac{1.03-1}{1.03^{\frac{1}{12}}-1} M = 12.1641194183M$. Therefore if she invests her whole salary in the plan, the accumulated value is $\frac{1680.59593496}{12.1641194183} S = 138.160098332S$. The percentage of salary she needs to invest is therefore $\frac{30.6995876861}{138.160098332} = 22.22\%$.

3. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
31	1.00000000000	40	1.46598394694	49	2.15275075482	58	3.16673115821
32	1.04333333333	41	1.52979439887	50	2.24688284912	59	3.30584525894
33	1.08856666667	42	1.59641582358	51	2.34518133040	60	3.45114616643
34	1.13578433333	43	1.66597358772	52	2.44783290930	61	3.60291246507
35	1.18507445667	44	1.73859871898	53	2.55503276764	62	3.76143543826
36	1.23652912243	45	1.81442816488	54	2.66698494742	63	3.92701965410
37	1.29024455921	46	1.89360506348	55	2.78390275783	64	4.09998357849
38	1.34632132704	47	1.97627902661	56	2.90600920129	65	4.28066021677
39	1.40486451487	48	2.06260643630	57	3.03353741915		

An employee aged 48 and 9 months has 21 years of service, and a current salary of $144,000 (for the coming year). He has a defined benefit pension plan with $\alpha = 0.015$ *and* S_{Fin} *is the average of his last 3 years' salary. The employee's mortality is given by* $\mu_x = 0.0000019(1.119)^x$. *The pension benefit is payable monthly in advance. The interest rate is* $i = 0.07$. *[This gives* $\ddot{a}_{65}^{(12)} = 12.0863903952$.*] Calculate the EPV of the accrued benefit under the assumption that the employee retires at age 65.*

We compute

$$s_{48\frac{9}{12}} = \frac{9}{12}s_{49} + \frac{3}{12}s_{48} = \frac{3}{4}\times 2.1527507548 + \frac{1}{4}\times 2.06260643630 = 2.13021467518$$

Therefore the final average salary is

$$144000\times\frac{3.76143543826 + 3.92701965410 + 4.09998357849}{3\times 2.13021467518} = \$265,628.184237$$

The EPV of the accrued benefit is therefore

$$265628.184237\times 0.015\times 21\times 12.0863903952(1.07)^{-16-\frac{3}{12}} = \$336,817.72$$

Standard Questions

4. *An employee aged 58 has been working with a company for 29 years. The employee's salary last year was $109,000. The salary scale is the same as for Question 3. The service table is given below:*

t	${}_tp^{(00)}$	1	2	3
0	*10000.00*	*42.04*	*0*	*10.26*
1	*9947.70*	*43.88*	*0*	*1.64*
2^-	*9902.18*		*1327.14*	
2	*8575.04*	*42.41*	*206.70*	*0.36*
3	*8325.57*	*44.24*	*334.93*	*1.15*
4	*7945.25*	*48.30*	*592.74*	*1.85*
5	*7302.36*	*55.26*	*950.64*	*1.61*
6	*6294.85*	*64.11*	*1366.20*	*0.44*
7^-	*4864.10*		*4864.10*	

Mortality after exiting the plan follows a Gompertz model with $B = 0.0000114$ *and* $C = 1.095$. *If the member withdraws, she receives a defered monthly pension starting from age 65, with 4% COLA. The death benefit of the plan is three times the employee's final average salary if the employee is still working at the time of death. If the employee has withdrawn, the death benefit is three times final average salary with COLA of 4%. The accrual rate for the pension is 0.02. The interest rate is* $i = 0.06$.

Calculate the EPV of the accrued benefit. [You may assume that events happen in the middle of each year.]

You are given the following values:

x	$\ddot{a}_x^{(12)}$
60	*14.09279*
60.5	*14.02004*
61.5	*13.87043*
62.5	*13.71527*
63.5	*13.55451*
64.5	*13.38806*
65	*13.3027*

We first calculate the EPV of accrued pension benefits for individuals who retire:

If the individual retires at age y, then their final average salary is $109000\frac{s_{y-3}+s_{y-2}+s_{y-1}}{3\times 3.16673115821} = 11473.4505451(s_{y-3}+s_{y-2}+s_{y-1})$

Retirement age	Probability	Final ave. Salary	EPV of pension at time of retirement	
60	0.132714	109067.926977	$0.58 * 109067.926977 * 14.09279 = 891501.40656$	$891501.40656 * 0.132714 * (1.06)^{-2} = 1$
60.5	0.020670	111463.633631	$0.58 * 111463.633631 * 14.02004 = 906380.269189$	$906380.269189 * 0.020670 * (1.06)^{-2.5} =$
61.5	0.033493	116361.592612	$0.58 * 116361.592612 * 13.87043 = 936111.488506$	$936111.488506 * 0.033493 * (1.06)^{-3.5} =$
62.5	0.059274	121477.440635	$0.58 * 121477.440635 * 13.71527 = 966335.620388$	$966335.620388 * 0.059274 * (1.06)^{-4.5} =$
63.5	0.095064	126820.991793	$0.58 * 126820.991793 * 13.55451 = 997017.912853$	$997017.912853 * 0.095064 * (1.06)^{-5.5} =$
64.5	0.136620	132402.507675	$0.58 * 132402.507675 * 13.38806 = 1028115.3758$	$1028115.3758 * 0.136620 * (1.06)^{-6.5} =$
65	0.486410	135254.068094	$0.58 * 135254.068094 * 13.3027 = 1043561.68915$	$1043561.68915 * 0.486410 * (1.06)^{-7} =$

so the total EPV of accrued pension benefits to retirees is $105299.677528 + 16195.1977895 + 25568.844162 + 44067.2379495 + 68791.817891 + 96176.2879696 + 337582.220219 = \$693,681.28351$.

Next we compute the EPV of accrued pension benefits to individuals who withdraw.

Age	Prob	Final ave. Salary (with COLA)	Prob. reach 65	EPV
58.5	0.001026	$102284.25 * (1.04)^{6.5} = 131985.27$	0.979793	$13.3027 * 0.979793 * 131985.27 * 0.58 * 0.001026 * (1.06)^{-7} = 680.82$
59.5	0.000164	$106774.18 * (1.04)^{5.5} = 132479.78$	0.982160	$13.3027 * 0.982160 * 132479.78 * 0.58 * 0.000164 * (1.06)^{-7} = 109.50$
60.5	0.000036	$111463.63 * (1.04)^{4.5} = 132979.05$	0.984758	$13.3027 * 0.984758 * 132979.05 * 0.58 * 0.000036 * (1.06)^{-7} = 24.19$
61.5	0.000115	$116361.59 * (1.04)^{3.5} = 133483.12$	0.987611	$13.3027 * 0.987611 * 133483.12 * 0.58 * 0.000115 * (1.06)^{-7} = 77.79$
62.5	0.000185	$121477.44 * (1.04)^{2.5} = 133992.03$	0.990744	$13.3027 * 0.990744 * 133992.03 * 0.58 * 0.000185 * (1.06)^{-7} = 126.02$
63.5	0.000161	$126820.99 * (1.04)^{1.5} = 134505.84$	0.994187	$13.3027 * 0.994187 * 134505.84 * 0.58 * 0.000161 * (1.06)^{-7} = 110.47$
64.5	0.000044	$132402.51 * (1.04)^{0.5} = 135024.59$	0.997970	$13.3027 * 0.997970 * 135024.59 * 0.58 * 0.000044 * (1.06)^{-7} = 30.42$

Thus the total EPV of accrued pension benefits to individuals who withdraw is $680.822734672 + 109.497061578 + 24.1903450287 + 77.7923409848 + 126.019871963 + 110.474415005 + 30.4235365833 = \$1,159.22030582$.

Next we calculate death benefits for individuals who die while employed:

Age at death	Probability	Final ave. Salary	EPV EPV
58.5	0.004204	102284.246384	$3 * 102284.246384 * 0.004204 * (1.06)^{-0.5} = 1252.96742183$
59.5	0.004388	106774.177360	$3 * 106774.177360 * 0.004388 * (1.06)^{-1.5} = 1287.9390462$
60.5	0.004241	111463.633631	$3 * 111463.633631 * 0.004241 * (1.06)^{-2.5} = 1225.90851227$
61.5	0.004424	116361.592612	$3 * 116361.592612 * 0.004424 * (1.06)^{-3.5} = 1259.43425471$
62.5	0.004830	121477.440635	$3 * 121477.440635 * 0.004830 * (1.06)^{-4.5} = 1354.21501609$
63.5	0.005526	126820.991793	$3 * 126820.991793 * 0.005526 * (1.06)^{-5.5} = 1525.95252283$
64.5	0.006411	132402.507675	$3 * 132402.507675 * 0.006411 * (1.06)^{-6.5} = 1743.63315305$

Thus the total EPV of death benefits for individuals who die while employed is

$1252.96742183+1287.9390462+1225.90851227+1259.43425471+1354.21501609+1525.95252283+1743.63315305 = \$9,650.04992698$.

Finally we need to compute the EPV of death benefits paid to individuals who withdraw. If an individual withdraws at age x with final average salary S, then the EPV of the death benefit is

$$S\int_x^{65} \mu_{y\,y-x}p_x(1.06)^{x-y}\,dy = S\int_x^{65} 0.0000114(1.095)^y e^{\frac{0.0000114}{\log(1.095)}(1.095^x-1.095^y)}(1.06)^{x-y}(1.04)^{y-x}\,dy$$

We compute this numerically:

Age at withdrawl	Final average salary	EPV of death benefit
58.5	102284.246384	5797.95375303
59.5	106774.177360	5402.70248165
60.5	111463.633631	4870.73256683
61.5	116361.592612	4176.7203626
62.5	121477.440635	3291.17736269
63.5	126820.991793	2179.77383813
64.5	132402.507675	802.548373246

Age at withdrawl	Probability	Conditional EPV of death benefit	EPV of death benefit
58.5	0.001026	5797.95375303	$0.001026 * 5797.95375303 = 5.94870055061$
59.5	0.000164	5402.70248165	$0.000164 * 5402.70248165 = 0.886043206991$
60.5	0.000036	4870.73256683	$0.000036 * 4870.73256683 = 0.175346372406$
61.5	0.000115	4176.7203626	$0.000115 * 4176.7203626 = 0.480322841699$
62.5	0.000185	3291.17736269	$0.000185 * 3291.17736269 = 0.608867812098$
63.5	0.000161	2179.77383813	$0.000161 * 2179.77383813 = 0.350943587939$
64.5	0.000044	802.548373246	$0.000044 * 802.548373246 = 0.0353121284228$

$5.94870055061 + 0.886043206991 + 0.175346372406 + 0.480322841699 + 0.608867812098 + 0.350943587939 + 0.0353121284228 = \$8.48553650017$.

Thus the total EPV of accrued benefits is

$693681.28351+1159.22030582+9650.04992698+8.48553650017 = \$704,499.04$.

5. *An individual aged 44 has 13 years of service, and last year's salary was \$56,000. The salary scale is* $s_y = 1.06^y$. *The accrual rate is 0.02. The interest rate is* $i = 0.05$. *There is no death benefit, and no exits other than death or retirement at age 65. The pension benefit is payable annually in advance. Mortality follows a Gompertz law with* $B = 0.0000047$ *and* $C = 1.132$. *You are given that* $\ddot{a}_{65}^{(12)} = 10.1197028436$. *Calculate this year's employer contribution to the plan using*

   (a) The projected unit method.

   The probability of the employee surviving to age 65 at the start of the year is $e^{-\frac{0.0000047}{\log(1.132)}\left((1.132)^{65}-(1.132)^{44}\right)} = 0.894931528226$. The probability of surviving to age 65 from the end of the year is $e^{-\frac{0.0000047}{\log(1.132)}\left((1.132)^{65}-(1.132)^{45}\right)} = 0.895980070866$. The individual's projected final average salary is $56000\frac{1.06^{19}+1.06^{20}+1.06^{21}}{3} = \$179,802.906731$. The current accrued pension benefit is therefore

   $$179802.906731\times13\times0.02\times10.1197028436\times0.894931528226\times(1.05)^{-21} = \$151,968.068723$$

   The accrued pension benefit if the life survives to the end of the year is

   $$179802.906731\times14\times0.02\times10.1197028436\times0.895980070866\times(1.05)^{-20} = \$172,042.152718$$

   The expected accrued value is

   $$172042.152718 \times \frac{0.894931528226}{0.895980070866}(1.05)^{-1} = \$163,657.920164$$

   The normal contribution is therefore $163657.920164 - 151968.068723 = \$11,689.851441$.

   (b) The traditional unit method.

   The individual's current final average salary is $56000\frac{1.06^{-2}+1.06^{-1}+1}{3} = \$52889.99644$. The current accrued pension benefit is therefore

   $$52889.99644\times13\times0.02\times10.1197028436\times0.894931528226\times(1.05)^{-21} = \$44,702.2284562$$

   Next year's final average salary is $56000\frac{1.06^{-1}+1+1.06}{3} = \$56,063.3962263$
   The accrued benefit end if is life of pension survives to year is

   $$56063.3962263\times14\times0.02\times10.1197028436\times0.895980070866\times(1.05)^{-20} = \$53,643.5564408$$

   The expected accrued value is

   $$53643.5564408 \times \frac{0.894931528226}{0.895980070866}(1.05)^{-1} = \$51,029.313099$$

   The normal contribution is therefore $51029.313099 - 44702.2284562 = \$6,327.0846428$.