# Introduction to Complex Numbers
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This handout is designed to cover the basic facts about complex numbers that will be needed for MATH 3090.

# 1 Types of Numbers

Probably the first type of numbers you learned about were numbers that represent the size of a set – i.e. the numbers $0, 1, 2, 3, \ldots$ These are called the *natural numbers*, and denoted $\mathbb{N}$. Later, you probably learned to add these numbers – for example, 2+3=5. This also led on to problems of the form $? + 2 = 5$. This one is easy to solve – $? = 3$ is the only solution. However, you were also able to create problems like $? + 5 = 2$. **This problem has no solution!**

At this point you might have simply moved on, happy in the knowledge that the equation $? + 5 = 2$ had no solution. However, at some point many years ago (possibly around the seventh century AD) a mathematician noticed that while the equation may have no solution, there are still a lot of things that we can say that any ? that were a solution would have to satisfy. For example, we could say $? + 3 = 0$, $? + 4 = 1$, $? + 6 = 3$, and so on. We could also say for instance that $2 \times ? + 6 = 0$, or that $? \times ? = 9$. Mathematicians therefore decided to give a name to this made-up number – they called it -3. When we take the set of all solutions to equations of the form $? + m = n$ for $m$ and $n$ natural numbers, we get the *integers*: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$, denoted $\mathbb{Z}$.

This same trick of inventing new solutions to problems that were previously insoluble was applied in other situations. For example, given the problem $3 \times ? = 2$, we invent a number $\frac{3}{2}$ to be a solution for ?. The set of numbers of the form $\frac{n}{m}$ where $n$ and $m$ are integers, and $m > 0$ is called the set of rational numbers, and denoted $\mathbb{Q}$. (Historically, rational numbers were invented before negative numbers, but from a certain logical point of view, it makes sense to discuss negative numbers first.)

$\mathbb{Q}$ is good for solving all equations involving just adding and multiplying by constants. (A collection of numbers for which we can solve all equations of the form $ax + b = c$ (where $a \neq 0$) is called a *field*.) We also extend the rationals by allowing limits of sequences, as we saw earlier in the course. This gives us the real numbers, $\mathbb{R}$.

However, even the real numbers are not sufficient for us to be able to solve all the equations that we might want to. For example, $x^2 = -1$ has no solution in the real numbers. To deal with this problem, we invent a new number, which we call $i$, with the property that $i^2 = -1$. Once we have invented this number, we naturally also invent all real multiples of it and all sums of real multiples of it with real numbers, i.e. we have all numbers of the form $a + bi$, where $a$ and $b$ are real numbers. This set is called the *complex numbers*, and is denoted $\mathbb{C}$. We call $a$ the *real part* of the complex number $a + bi$, and we call $b$ the *imaginary part*. Adding, subtracting and multiplying complex numbers is straightforward:

$(a + bi) + (c + di) = a + c + (b + d)i$, $(a + bi) - (c + di) = a - c + (b - d)i$, and $(a + bi)(c + di) = ac + bci + adi + bd(i^2) = (ac - bd) + (ad + bc)i$. Dividing one complex number by another is more tricky, and involves an important idea – the complex conjugate.

**Definition 1.1.** Given a complex number $z = a + bi$, its complex conjugate is the number $\overline{z} = a - bi$.

$\overline{z}$ has the important property that $z\overline{z} = a^2 + b^2$ and $z + \overline{z} = 2a$ are both real numbers. This means that it can be used to simplify a lot of expressions involving complex numbers. For example, given a fraction $\frac{a+bi}{c+di}$, we can multiply both the numerator and the denominator by $c - di$, to get

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c + di)}{c^2 + d^2} = \frac{ac - bd}{c^2 + d^2} + \frac{ad + bc}{c^2 + d^2}i$$

which is the form we used above for complex numbers.

# 2 Geometry of Complex Numbers

We can think of real numbers as being distances (in a specified direction) along a line. Complex numbers have two real numbers associated with them – the real and imaginary parts. We can therefore associate the complex number $a + bi$ with the point $(a, b)$ in 2-dimensional space. This description of complex numbers is called the Argand plane. Note that in this plane, the real numbers are exactly the ones on the horizontal axis (which is therefore called the real line). The vertical axis is called the imaginary axis, because all the numbers on it are imaginary. In this plane, the conjugate of a complex number is its reflection in the real axis.

This geometric point of view allows us to describe $a + bi$ in another way – give the polar coordinates of the point $(a, b)$. This gives us the following definitions:

**Definition 2.1.** The *modulus* $|z|$ of the complex number $z = a + bi$ is the distance of the point $(a, b)$ from the origin. It is given by $|z| = \sqrt{(a^2 + b^2)}$.

**Definition 2.2.** The *argument* of the non-zero complex number $z = a + bi$ is the anticlockwise angle from the positive real axis to the line segment between the origin and the point $(a, b)$. It is the value of $\theta$ such that $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, while $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$. There are several different values of $\theta$ that satisfy this – in particular, if $\theta$ is one possible value for the argument of $z$, then $\theta + 2\pi$ is another. The argument of $z$ is generally chosen to lie in some particular range, usually either $(-\pi, \pi]$ or $[0, 2\pi)$.

For example $i$ has modulus 1, and argument $\frac{\pi}{2}$, $-2i$ has modulus 2 and argument $-\frac{\pi}{2}$, and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ has modulus 1 and argument $\frac{\pi}{3}$.

The modulus-argument form for complex numbers is not helpful when we add two complex numbers – we can say little about how the modulus and argument

of the sum depend on the modulus and argument of the numbers being added. One important thing that we can say about the modulus of a sum is the triangle inequality.

**Lemma 2.3.** *If $z$ and $w$ are complex numbers, then $|z + w| \leqslant |z| + |w|$ and $|z - w| \geqslant |z| - |w|$.*

*Proof.* Let $z = a + bi$ and $w = c + di$. Then $|z + w|^2 = (a+c)^2 + (b+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ac + 2bd$, while $(|z| + |w|)^2 = a^2 + b^2 + c^2 + d^2 + 2\sqrt{(a^2 + b^2)(c^2 + d^2)}$. However, $(ac + bd)^2 = a^2c^2 + b^2d^2 + 2abcd$, and $abcd \leqslant \frac{a^2d^2 + b^2d^2}{2}$, as $a^2d^2 + b^2c^2 - 2abcd = (ad - bc)^2 \geqslant 0$. Therefore, $2ac + 2bd \leqslant 2\sqrt{(a^2 + b^2)(c^2 + d^2)}$, sice the latter term is non-negative. Therefore, $|z + w|^2 \leqslant (|z| + |w|)^2$. Since both are positive, we can take square roots to get $|z + w| \leqslant |z| + |w|$.

To get $|z - w| \geqslant |z| - |w|$, apply $|z + w| \leqslant |z| + |w|$ to $z - w$ and $w$. $\qquad \square$

Note that equality only holds here if one number is a positive real multiple of the other.

However, we can express the modulus and argument of a product directly in terms of the moduli and arguments of the numbers being multiplied. In fact:

**Lemma 2.4.** $|zw| = |z||w|$, *while* $\arg(zw) = \arg(z) + \arg(w)$.

We will prove this in the next section, when it will follow from properties of the function $e^z$.

# 3   Sequences and Series of Complex Numbers

Recall that when we defined convergence of sequences of real numbers, all we needed was a notion of distance between two numbers. Since complex numbers correspond to points in a plane, we have a natural notion of distance between them – the distance from $z$ to $w$ is given by $|z - w|$. This allows us to extend the definition of convergence of sequences and series to sequences and series of complex numbers.

**Definition 3.1.** A sequence $z_n$ of complex numbers converges to a complex number $z$ if $(\forall \epsilon > 0)(\exists N)(\forall n > n)(|z - z_n| < \epsilon)$.

Most of the proofs in the course so far work just as well for complex numbers as for real ones. For example, if a series $\sum_{n=0}^{\infty} z_n$ of complex numbers is such that $\sum_{n=0}^{\infty} |z_n|$ converges, then $\sum_{n=0}^{\infty} z_n$ converges. This means that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for any complex number $z$ with $|z| < R$, where $R$ is the radius of convergence of the series as a series of real numbers.

This explains why some power series have a smaller radius of convergence than one might expect from looking at just their real part. For example, the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$ (see Homework Sheet 5 question 1) corresponds to the function $\frac{1}{x^2 + 4}$, which is infinitely differentiable at every real number. However, $\frac{1}{x^2 + 4}$ is not defined at $x = 2i$, so the radius of convergence is only 2.

This means that we can extend any function on the real numbers that has a Taylor series to a function on the complex numbers. An important example of this is the function $e^x$. We get a function on the whole complex numbers given by $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. The identity $e^{x+y} = e^x e^y$ holds for all complex numbers, so we only need to work out the value of $e^{i\alpha}$ for every real $\alpha$, to be able to calculate $e^z$ for any complex number $z$. For $z = i\alpha$, we get $e^{i\alpha} = 1 + i\alpha - \frac{\alpha^2}{2} - \frac{i\alpha^3}{3!} + \ldots$. Since the series is absolutely convergent, we can separate the real and imaginary terms, to get

$$e^{i\alpha} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\right)$$

We recognise the two series here as the Taylor series for $\cos\alpha$ and $\sin\alpha$ respectively. Therefore, we have

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

This is a very important formula, which gives us easy proofs of a large number of trigonometric identities. For example, we can now prove Lemma 6.26 simply by summing a geometric series, and taking real and imaginary parts.

The formula $e^{i\alpha} = \cos\alpha + i\sin\alpha$ also explains why the modulus and argument of complex numbers behave the way they do when we multiply two complex numbers. Note that $|e^{i\alpha}|^2 = \cos^2\alpha + \sin^2\alpha = 1$ for any $\alpha$. Therefore, $e^{i\alpha}$ is a complex number with modulus 1 and argument $\alpha$. Given a non-zero complex number $z$, we can say $z = |z|\frac{z}{|z|}$. $|z|$ is a positive real number, while $\frac{z}{|z|}$ has modulus 1. Therefore, we can write $|z| = e^x$ for some real $x$, and $\frac{z}{|z|} = e^{i\alpha}$ for some real $\alpha$ in the interval $[0, 2\pi)$. Thus $z = e^{x+i\alpha}$. If we have $w = e^{y+i\beta}$, then their product is given by $zw = e^{x+y+i(\alpha+\beta)}$, so $|zw| = e^{x+y} = |z||w|$, and $\arg(zw) = \alpha + \beta = \arg(z) + \arg(w)$ (we might need to add or subtract $2\pi$ to get a value in the correct range for $\arg(zw)$).