I don't think I presented this proof very well in the lectures, so I've written it out more clearly (hopefully) here.

Theorem 1. *If A is a finite set, then there is no bijection from A to any proper subset A' of A.*

Proof. We begin with a special case:

Lemma 1. *There is no bijection $f : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, m-1\}$ when $m \neq n$.*

Proof. Since a bijection from $\{0, 1, \ldots, n-1\}$ to $\{0, 1, \ldots, m-1\}$ gives a bijection from $\{0, 1, \ldots, m-1\}$ to $\{0, 1, \ldots, n-1\}$, we may assume that $m < n$. When $n = 0$, there is no $m < n$, so the lemma is vacuously true. Suppose the lemma holds for all smaller values of n.

Suppose $f : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, m-1\}$ is a bijection, and $m < n$. Let $k = f(n-1)$, and $l = f^{-1}(m-1)$, and define $g : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, m-2\}$ by $g(x) = \begin{cases} f(x) & \text{if } x \neq l \\ k & \text{if } x = l \end{cases}$. g is an injection, since if $g(x) = g(y)$, then either $f(x) = f(y)$, or $x = l$ or $y = l$, but if $x = l$, then $g(x) = f(n-1)$, so that $g(y) = g(x)$ only occurs for $y = l$. Therefore, g is a bijection from $\{0, 1, \ldots, n-2\}$ to $\{0, 1, \ldots, m-2\}$. This can't happen by our inductive hypothesis, so we can't have an injection from $\{0, 1, \ldots, n-1\}$ to $\{0, 1, \ldots, m-1\}$ for any $m < n$. Therefore, the theorem holds by induction. □

A is finite, so there is a bijection f from A to $\{0, 1, \ldots, n-1\}$ for some natural number n.

The subset A' is finite, since the function $g : A' \to \{0, 1, \ldots, m-1\}$ sending a to the number of elements $x \in A'$ such that $f(x) < f(a)$ (this function is well defined by the lemma) is a bijection for some value of m (Which will be the size of the subset A'). Therefore, if we have a bijection $h : A \to A'$, then we can form a bijection $g \circ h \circ f^{-1} : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, m-1\}$, which does not exist by the lemma. Therefore, by contradiction, there can't be a bijection $h : A \to A'$. □