MATH 3090, Advanced Calculus I
Fall 2006
Final Examination
Model Solutions

1 *Which of the following series of functions converge uniformly on the interval (0,1)?*
(a) $\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$

As $x > 0$, $\frac{1}{(x+n)^2} < \frac{1}{n^2}$, so this series converges uniformly by the Weierstrass M-test with $M_n = \frac{1}{n^2}$.

(b) $\sum_{n=1}^{\infty} \frac{1}{(x+1)^n}$

The error in the Nth partial sum is $\sum_{n=N+1}^{\infty} \frac{1}{(x+1)^n} = \frac{1}{(x+1)^{N+1}} \times \frac{1}{1-\frac{1}{x+1}} = \frac{1}{x(x+1)^N}$. If we choose $\epsilon = \frac{1}{2}$, given any N, we can choose $x = 2^{\frac{1}{N}} - 1$ (so that $(x+1)^N = 2$). Then $\frac{1}{x(x+1)^N} \geqslant \frac{1}{2}$ ($x < 1$, so $\frac{1}{x} > 1$). Therefore, the series does not converge uniformly.

2 *Which of the following series converge? For series which converge, is the convergence absolute? Justify your answers. (You may assume convergence of geometric series and* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *for* $p > 1$*, and divergence of* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *for* $p \leqslant 1$*.)*
(a) $\sum_{n=0}^{\infty} (-1)^n \frac{3}{2n+5}$

$\frac{3}{2n+5}$ is a decreasing function of n, so by the alternating series test, the series converges. For $n \geqslant 5$, $\frac{3}{2n+5} \geqslant \frac{1}{n}$, so the series does not converge absolutely by comparison to $\sum_{n=5}^{\infty} \frac{1}{n}$.

(b) $\sum_{n=0}^{\infty} \frac{1\times 4\times 7\times\cdots\times(3n+1)}{2\times5\times8\times\cdots\times(3n+5)}$

$1 \times 4 \times 7 \times \cdots \times (3n+1) = 3^{n+1}\frac{\Gamma\left(n+\frac{4}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}$, while $2 \times 4 \times 8 \times \cdots \times (3n+5) = 3^{n+2}\frac{\Gamma\left(n+\frac{8}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$. Therefore, the series is:

$$\frac{\Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{4}{3}\right)}{\Gamma\left(n+\frac{8}{3}\right)}$$

and we know that $\frac{\Gamma(x)x^{\alpha}}{\Gamma(x+\alpha)} \to 1$ as $x \to \infty$, so taking $x = \left(n+\frac{4}{3}\right)$ and $\alpha = \frac{4}{3}$, for large n, the series is approximately $\sum \frac{1}{\left(n+\frac{4}{3}\right)^{\frac{4}{3}}}$, which converges by comparison to $\sum \frac{1}{n^{\frac{4}{3}}}$ ($\frac{4}{3} > 1$). Since all the terms are positive, the convergence is absolute.

3 *Prove that the ratio test works – i.e. show that if a_n is a sequence of positive real numbers, and $\frac{a_{n+1}}{a_n} \to x < 1$ as $n \to \infty$, then $\sum_{n=0}^{\infty} a_n$ converges. [You may assume the comparison test and convergence of geometric series.]*

Theorem 1. *If a_n is a sequence of positive real numbers, and $\frac{a_{n+1}}{a_n} \to x < 1$ as $n \to \infty$, then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. Since $x < 1$, there is an $\epsilon > 0$ such that $x+\epsilon < 1$. As $\frac{a_{n+1}}{a_n} \to x$, we can choose N so that for $n > N$, $\left|\frac{a_{n+1}}{a_n} - x\right| < \epsilon$, and thus $\frac{a_{n+1}}{a_n} < x + \epsilon$. This means that $a_{N+1} < (x+\epsilon)a_N$, $a_{N+2} < (x+\epsilon)a_{N+1} < (x+\epsilon)^2 a_N$, and so on. In general, $a_{N+k} < (x+\epsilon)^k a_N$ (We can prove this by induction). However, the series $\sum_{n=k}^{\infty} a_N(x+\epsilon)^k$ is a geometric series, with common ratio less than 1, so it converges. Therefore, $\sum_{n=N}^{\infty} a_n$ converges by the comparison test. Also, $\sum_{n=0}^{N-1} a_n$ is a finite sum, so it converges. Therefore, $\sum_{n=0}^{\infty} a_n$ converges. □

4 *Find the radius of convergence of the following series.*

(a) $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n(n+3)}$

The ratio $\frac{a_{n+1}}{a_n}$ of consecutive terms in this series is $\frac{x^{2n+2}4^n(n+3)}{x^{2n}4^{n+1}(n+4)} = \frac{x^2(n+3)}{4(n+4)}$. However, as $n \to \infty$, $\frac{n+3}{n+4} \to 1$, so the ratio of consecutive terms $\frac{a_{n+1}}{a_n} \to \frac{x^2}{4}$ as $n \to \infty$. Therefore, by the ratio test, the series converges whenever $|x| < 2$, and diverges whenever $|x| > 2$. Thus the radius of convergence is 2.

(b) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

The ratio $\frac{a_{n+1}}{a_n}$ of consecutive terms in this series is $\frac{2^{n+1}x^{n+1}n!}{2^n x^n(n+1)!} = \frac{2x}{n+1}$. For any value of x, $\frac{2x}{n+1} \to 0$ as $n \to \infty$. Therefore, the series converges for all x, so its radius of convergence is infinite.

5 *Find the Fourier series for the following functions: [You may use either the $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ or the $\frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ form for the Fourier series]*

(a) $f(x) = x^2$ *for* $0 \leqslant x < 2\pi$, *and* f 2π*-periodic.*

$\sum_{n=-\infty}^{\infty} c_n e^{inx}$

The coefficients c_n are given by $c_n = \frac{1}{2\pi}\int_0^{2\pi} x^2 e^{-inx} dx$. For $n = 0$, we get $c_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{4\pi^2}{3}$. For $n \neq 0$, we integrate by parts:

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_0^{2\pi} x^2 e^{-inx} dx = \frac{1}{2\pi}\left(\left[\frac{x^2 e^{-inx}}{-in}\right]_0^{2\pi} - \int_0^{2\pi} \frac{2xe^{-inx}}{-in} dx\right) \\
&= \frac{1}{2\pi}\left(\frac{4\pi^2}{-in} - \left[\frac{2xe^{-inx}}{-n^2}\right]_0^{2\pi} + \int_0^{2\pi} \frac{2e^{-inx}}{-n^2} dx\right) \\
&= \frac{1}{2\pi}\left(\frac{4\pi^2}{-in} + \frac{4\pi}{n^2}\right) = \frac{2}{n^2} + \frac{2\pi i}{n}
\end{aligned}
$$

Therefore, the Fourier series is $f(x) = \frac{4\pi^2}{3} + \sum_{n\neq 0}\left(\frac{2}{n^2} + \frac{2\pi i}{n}\right) e^{inx}$.

$\frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

The calculation of a_0 is similar to that of c_0, and gives $a_0 = \frac{8\pi^2}{3}$. The other coefficients are given by

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx) dx \\
&= \frac{1}{\pi}\left(\left[\frac{x^2 \sin(nx)}{n}\right]_0^{2\pi} - \int_0^{2\pi} \frac{2x\sin(nx)}{n} dx\right) \\
&= \frac{1}{\pi}\left(\left[\frac{2x\cos(nx)}{n^2}\right]_0^{2\pi} - \int_0^{2\pi} \frac{2\cos(nx)}{n^2} dx\right) \\
&= \frac{4}{n^2}
\end{aligned}
$$

And

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx) dx \\
&= \frac{1}{\pi}\left(\left[-\frac{x^2 \cos(nx)}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{2x\cos(nx)}{n} dx\right) \\
&= \frac{1}{\pi}\left(-\frac{4\pi^2}{n} + \left[\frac{2x\sin(nx)}{n^2}\right]_0^{2\pi} - \int_0^{2\pi} \frac{2\sin(nx)}{n^2} dx\right) \\
&= -\frac{4\pi}{n}
\end{aligned}
$$

So the Fourier series is $f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos(nx)}{n^2} - \frac{4\pi\sin(nx)}{n}$.

(b) $f(x) = \begin{cases} 1 & \textit{if } -\pi < x \leqslant -\frac{\pi}{2} \\ -\frac{1}{2} & \textit{if } -\frac{\pi}{2} < x \leqslant \pi \end{cases}$ *and f 2π-periodic.*

$\sum_{n=-\infty}^{\infty} c_n e^{inx}$
The coefficients c_n for $n \neq 0$ are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx =$$
$$\frac{1}{2\pi} \left(\int_{-\pi}^{-\frac{\pi}{2}} e^{-inx} dx + \frac{1}{2} \int_{-\frac{\pi}{2}}^{\pi} -e^{-inx} dx \right)$$
$$= \frac{1}{2\pi} \left(\frac{i^n - (-1)^n}{-in} - \frac{(-1)^n - i^n}{-2in} \right) = \frac{3(i^{n+1} - i^{2n+1})}{4\pi n}$$

and $c_0 = \frac{1}{2\pi} \left(\int_{-\pi}^{-\frac{\pi}{2}} 1 dx + \int_{-\frac{\pi}{2}}^{\pi} -\frac{1}{2} dx \right) = -\frac{1}{8}$. Therefore, the Fourier series is $f(x) = -\frac{1}{8} + \frac{3}{4} \sum_{n \neq 0} \frac{i^{n+1} - i^{2n+1}}{\pi n} e^{inx}$.

$\frac{1}{2} a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$
From the above calculation, we get $a_0 = -\frac{1}{4}$, while:

$$a_n = \frac{1}{2\pi} \left(\int_{-\pi}^{-\frac{\pi}{2}} \cos(nx) dx + \frac{1}{2} \int_{-\frac{\pi}{2}}^{\pi} -\cos(nx) dx \right) = \frac{1}{2\pi} \left(\frac{\sin(-\frac{n}{2}\pi)}{n} + \frac{\sin(-\frac{n}{2}\pi)}{2n} \right)$$
$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-3}{4\pi n} & \text{if } n = 4k+1 \text{ for some integer } k \\ \frac{3}{4\pi n} & \text{if } n = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$b_n = \frac{1}{2\pi} \left(\int_{-\pi}^{-\frac{\pi}{2}} \sin(nx) dx + \frac{1}{2} \int_{-\frac{\pi}{2}}^{\pi} -\sin(nx) dx \right)$$
$$= \frac{1}{2\pi} \left(\frac{(-1)^n - \cos(-\frac{n}{2}\pi)}{n} + \frac{(-1)^n - \cos(-\frac{n}{2}\pi)}{2n} \right)$$
$$= \begin{cases} \frac{3}{2\pi n} & \text{if } n \text{ is even but not divisible by } 4 \\ 0 & \text{if } n \text{ is divisible by } 4 \\ \frac{3}{4\pi n} & \text{if } n \text{ is odd} \end{cases}$$

Therefore, the Fourier series is $f(x) = -\frac{1}{8} + \frac{3}{4} \sum_{k=0}^{\infty} \frac{-\cos((4k+1)x) - \sin((4k+1)x)}{\pi(4k+1)} + \frac{-2\sin((4k+2)x)}{\pi(4k+2)} + \frac{\cos((4k+3)x) - \sin(4k+3)x}{\pi(4k+3)}$.

(c) $f(x) = \begin{cases} 4 & \textit{if } -\frac{\pi}{2} < x \leqslant 0 \\ 1 & \textit{if } -\pi < x \leqslant -\frac{\pi}{2} \textit{ or } 0 < x \leqslant \pi \end{cases}$ *and f 2π-periodic.*

This f is obtained from the f in (b) by multiplying by 2, adding 2, and making the change of variable $x \mapsto x+\frac{\pi}{2}$. i.e. If $g(x) = \begin{cases} 1 & \text{if } -\pi < x \leqslant -\frac{\pi}{2} \\ -\frac{1}{2} & \text{if } -\frac{\pi}{2} < x \leqslant \pi \end{cases}$ and g is 2π-periodic, then $f(x) = 2g(x - \frac{\pi}{2}) + 2$, so its Fourier series is: $f(x) = \frac{7}{4} + \sum_{n\neq 0} \frac{3(i^{n+1}-i^{2n+1})}{2\pi n} e^{in\left(x-\frac{\pi}{2}\right)} = \frac{7}{4} + \sum_{n\neq 0} \frac{3(i-i^{n+1})}{2\pi n} e^{inx}$. Alternatively,

$$f(x) = \frac{7}{4} + \frac{3}{2}\sum_{k=0}^{\infty} \frac{-\cos\left((4k+1)\left(x-\frac{\pi}{2}\right)\right) - \sin\left((4k+1)\left(x-\frac{\pi}{2}\right)\right)}{\pi(4k+1)}$$

$$+\frac{-2\sin\left((4k+2)\left(x-\frac{\pi}{2}\right)\right)}{\pi(4k+2)} + \frac{\cos\left((4k+3)\left(x-\frac{\pi}{2}\right)\right) - \sin\left((4k+3)\left(x-\frac{\pi}{2}\right)\right)}{\pi(4k+3)}$$

$$= \frac{7}{4} + \frac{3}{2}\sum_{k=0}^{\infty} \frac{-\sin((4k+1)x) + \cos((4k+1)x)}{\pi(4k+1)} + \frac{2\sin((4k+2)x)}{\pi(4k+2)}$$

$$+\frac{-\sin((4k+3)x) - \cos((4k+3)x)}{\pi(4k+3)}$$

6 *Find the Fourier sine series for the following functions on the interval* $[0, \pi]$.

(a) $f(x) = \sin^2 x$

The coefficients b_n are given by:

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos(2x)}{2}\sin(nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi}\left(\sin(nx) - \frac{\sin((n+2)x)+\sin((n-2)x)}{2}\right)dx \\ &= \begin{cases} \frac{1}{\pi}\left(\frac{2}{n} - \frac{1}{n+2} - \frac{1}{n-2}\right) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Therefore, the Fourier sine series is

$$f(x) = \sum_{m=0}^{\infty} \frac{-8}{\pi(2m+1)((2m+1)^2-4)}\sin((2m+1)x)$$

.

(b) $f(x) = x(\pi - x)$

The Fourier coefficients are given by:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin(nx)dx = \frac{2}{\pi}\left(\left[\frac{-x(\pi-x)\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{(\pi-2x)\cos(nx)}{n}dx\right)$$

$$= \frac{2}{\pi}\left(\left[\frac{(\pi-2x)\sin(nx)}{n^2}\right]_0^{\pi} - \int_0^{\pi}\frac{-2\sin(nx)}{n^2}dx\right) = \frac{2}{\pi}\left[\frac{-2\cos(nx)}{n^3}\right]_0^{\pi}$$
$$= \frac{4(1-(-1)^n)}{\pi n^3} = \begin{cases} \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}$$

So the sine series is $f(x) = \sum_{m=0}^{\infty} \frac{8\sin((2m+1)x)}{\pi(2m+1)^3}$.

7 *Given that $f(x) = x$ on $[-\pi, \pi]$, extended to a 2π-periodic function, has Fourier series $f(x) = \sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{inx}$, use Parseval's identity to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.*

Parseval's identity: $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$.

This gives $\sum_{n\neq 0} \frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{6\pi} = \frac{\pi^2}{3}$.

However, in this sum, we have counted each $\frac{1}{n^2}$ twice – once for n and once for $-n$, so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

8 *A guitar string of length 1 is plucked at the point one third of the way along its length. When it is plucked, the displacement is therefore given by $u(x,0) = \begin{cases} \frac{x}{2} & \text{if } x \leqslant \frac{1}{3} \\ \frac{1-x}{4} & \text{if } \frac{1}{3} < x \leqslant 1 \end{cases}$. It is then released from rest in this position (so $\frac{\partial u}{\partial t}(x,0) = 0$). Use separation of variables and Fourier series to find $u(x,t)$ for $t > 0$. (The ends of the string are fixed, so $u(0,t) = u(1,t) = 0$. You may assume the string satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.) [2 marks]*

We first look for solutions of the form $u(x,t) = \Theta(x)\Phi(t)$ that satisfy the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, and the boundary conditions $u(0,t) = u(1,t) = 0$. When $u(x,t) = \Theta(x)\Phi(t)$, the wave equation becomes $\Theta(x)\ddot{\Phi}(t) = c^2\Theta''(x)\Phi(t)$, so $\frac{\ddot{\Phi}(t)}{\Phi(t)} = c^2\frac{\Theta''(x)}{\Theta(x)}$. The left-hand side depends only on t, while the right-hand side depends only on x, so both must equal some constant λ. To get $u(0,t) = u(1,t) = 0$, we must have $\Theta(0) = \Theta(1) = 0$, and Θ satisfies $\Theta''(x) - \frac{\lambda}{c^2}\Theta(x) = 0$. To get $\Theta(0) = \Theta(1) = 0$, we must have $\Theta(x) = \sin(n\pi x)$ for some integer n. For this to be a solution to $\Theta''(x) - \frac{\lambda}{c^2}\Theta(x) = 0$, we must have that $\lambda = -n^2\pi^2c^2$. We now solve for Φ, to get $\Phi(t) = a\cos(n\pi ct) + b\sin(n\pi ct)$ for some a and b. However, because $\frac{\partial u}{\partial t}(x,0) = 0$, we must have $\dot{\Phi}(0) = 0$, and therefore, $b = 0$.

We therefore have solutions of the form $a\sin(n\pi x)\cos(n\pi ct)$ for n an integer. (In fact, we may assume that n is a positive integer, since sine and cosine are odd and even respectively, so the function for $-n$ is just -1 times the function for n.) Our general solution is just a sum of these:

$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi ct)$. Now, using our initial condition, we can find the a_n as the coefficients of the Fourier sine series for the initial displacement.

$$
\begin{aligned}
a_n &= 2\int_0^1 u(x,0)\sin(n\pi x)dx = 2\left(\int_0^{\frac{1}{3}} \frac{x\sin(n\pi x)}{2}dx + \int_{\frac{1}{3}}^1 \frac{(1-x)\sin(n\pi x)}{4}dx\right) \\
&= \left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^{\frac{1}{3}} + \int_0^{\frac{1}{3}} \frac{\cos(n\pi x)}{n\pi}dx + \left[-\frac{(1-x)\cos(n\pi x)}{2n\pi}\right]_{\frac{1}{3}}^1 - \int_{\frac{1}{3}}^1 \frac{\cos(n\pi x)}{n\pi}dx \\
&= -\frac{\cos\left(\frac{n\pi}{3}\right)}{3n\pi} + \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2} + \frac{\cos\left(\frac{n\pi}{3}\right)}{3n\pi} + \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2} \\
&= 2\frac{\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2} = \begin{cases} 0 & \text{if } n = 3m \text{ for some integer } m \\ \frac{(-1)^m\sqrt{3}}{(3m+1)^2\pi^2} & \text{if } n = 3m+1 \text{ for some integer } m \\ \frac{(-1)^m\sqrt{3}}{(3m+2)^2\pi^2} & \text{if } n = 3m+2 \text{ for some integer } m \end{cases}
\end{aligned}
$$

Therefore, the string satisfies $u(x,t) = \sum_{m=0}^{\infty} \frac{(-1)^m\sqrt{3}\sin((3m+1)\pi x)\cos((3m+1)\pi ct)}{(3m+1)^2\pi^2} + \frac{(-1)^m\sqrt{3}\sin((3m+2)\pi x)\cos((3m+2)\pi ct)}{(3m+2)^2\pi^2}$ for $t > 0$.

9 *A metal rod of length π, satisfying the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, where $u(x,t)$ is the temperature of the point a distance x along the rod at time t, is heated to a uniform $50°$ C, then one end of the rod is fixed at $0°$ C, so that $u(0,t) = 0$ for all t, and the other end is insulated, so that $\frac{\partial u}{\partial x}(\pi,t) = 0$ for all t.*

(a) Use separation of variables to find a family of solutions $u(x,t)$, that can be expressed as $\Theta(x)\Phi(t)$, that satisfy the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(\pi,t) = 0$ for all t. [Hint: to satisfy the boundary conditions, you should get $\Theta(x) = c\sin\left(\left(n+\frac{1}{2}\right)x\right)$ for constant c.]

We look for a solution of the form $u(x,t) = \Theta(x)\Phi(t)$. The heat equation then becomes $\Theta(x)\dot{\Phi}(t) = k\Theta''(x)\Phi(t)$, and so $\frac{\dot{\Phi}(t)}{\Phi(t)} = k\frac{\Theta''(x)}{\Theta(x)}$. The left-hand side depends only on t, while the right-hand side depends only on x, so they must both be equal to some constant λ. The boundary conditions, $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(\pi,t) = 0$ for all t, mean that $\Theta(0) = 0$ and $\Theta'(\pi) = 0$. To get this, we must have $\Theta(x) = a\sin\left(\left(n+\frac{1}{2}\right)x\right)$, and so $\lambda = -\left(n+\frac{1}{2}\right)^2 k$. Therefore, the solutions are

$$u_n(x,t) = a_n \sin\left(\left(n+\frac{1}{2}\right)x\right)e^{-\left(n+\frac{1}{2}\right)^2 kt}$$

Use Fourier series to find a solution for $u(x,t)$ that satisfies the boundary conditions and the intial condition $u(x,0) = 50$ for all x. [Hint: since you're trying to get a series with terms $\sin\left(\left(n+\frac{1}{2}\right)x\right)$, you will need to extend $u(x,0)$ to a function on $[0,2\pi]$. To get only the odd terms, you should make the extension symmetric about π (so $u(2\pi - x, 0) = u(x,0)$).]

We have that $u(x,t) = \sum_{n=0}^{\infty} a_n \sin\left(\left(n+\frac{1}{2}\right)x\right) e^{-\left(n+\frac{1}{2}\right)^2 kt}$, and $u(x,0) = 50$ for all x. We need to express $u(x,0)$ as a sum of functions of the form $a_n \sin\left(\left(n+\frac{1}{2}\right)x\right)$. To do this, we see that if we extend f to a function on the interval $[0,2\pi]$, then its sine series will have terms of the form $a_n \sin\left(\frac{m}{2}x\right)$ for a natural number m. We need to ensure that the terms for even n are all zero. This can be done by extending f in such a way that it is symmetric about π, since $\sin(kx)$ satisfies $\sin(k(2\pi - x)) = -\sin(kx)$ for any integer k, so if f is symmetric about π, then $\int_0^{2\pi} f(x)\sin(kx)dx = 0$. On the other hand, $\int_0^{2\pi} f(x)\sin\left(\left(n+\frac{1}{2}\right)x\right)dx = 2\int_0^{\pi} f(x)\sin\left(\left(n+\frac{1}{2}\right)x\right)dx$, so the Fourier coefficients are:

$$a_n = \frac{2}{\pi}\int_0^{\pi} 50\sin\left(\left(n+\frac{1}{2}\right)x\right) = \frac{2}{\pi}\left[\frac{-50\cos\left(\left(n+\frac{1}{2}\right)x\right)}{n+\frac{1}{2}}\right]_0^{\pi} = \frac{100}{\left(n+\frac{1}{2}\right)\pi}$$

Therefore, the solution is $u(x,t) = \sum_{n=0}^{\infty} \frac{100\cos\left(\left(n+\frac{1}{2}\right)x\right)e^{-\left(n+\frac{1}{2}\right)^2 kt}}{\left(n+\frac{1}{2}\right)\pi}$.