MATH 3090, Advanced Calculus I
Fall 2006
Toby Kenney
Homework Sheet 1
Due in: Monday 18th September, 11:30 AM

On this sheet, all sequences are sequences of real numbers. Please hand in solutions to questions 1-3. Question 4 is for interest only – feel free to collaborate on it or ask me about it.

Compulsory questions

1 Prove from the definition of convergence that the sequence $1, 2, 3, \ldots$ does not converge to any real number x.

2 (a) Show that if (x_n) is a sequence, such that every subsequence (x_{n_i}) has a subsequence which converges to x, then $x_n \to x$. [Hint: Suppose x_n does not converge to x. Then there is some *epsilon* > 0 such that for every N, there is $n > N$ with $|x_n - x| > \epsilon$. Construct a sequence of these x_n. does it have a subsequence which converges to x?]

(b) Deduce that if y_n is a bounded sequence that does not converge, then it has (at least) two convergent subsequences which converge to different limits. [Hint: If x_n does not converge to x, then as in part (a), we can construct a subsequence that has no subsequence converging to x. Use Bolzano-Weierstrass on this subsequence.]

3 Which of the following series converge and which diverge? Justify your answers. (You may assume convergence and divergence of the series covered in lectures.)
(a) $\sum_{n=0}^{\infty} \frac{3^n}{n!}$
(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
(c) $\sum_{n=1}^{\infty} \sqrt{n^2+1} - n$ [Hint: $x^2 - y^2 = (x+y)(x-y)$]
(d) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ [Hint: to integrate $\frac{1}{x \log x}$, you may find the substitution $u = \log x$ helpful.]

Optional questions

4 (a)Construct a sequence that has subsequences converging to every $0 \leqslant x \leqslant 1$ (Hint: every real number between 0 and 1 can be expressed as a decimal).

(b) Is it possible to construct a sequence that has subsequences converging to every $0 < x \leqslant 1$, but no subsequence converging to 0? Either give the sequence or give a proof that no such sequence exists.