MATH 3090, Advanced Calculus I
Fall 2006
Final Examination
Mock

1 Which of the following series of functions converge uniformly on the interval (0,1)? Justify your answers.

(a) $\sum_{n=1}^{\infty} x^n$

(b) $\sum_{n=1}^{\infty} x^n(1-x)^2$

2 Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence?

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{n \log n}$

3 Which of the following series converge? For series which converge, is the convergence absolute? Justify your answers. (You may assume convergence of geometric series and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$, and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \leqslant 1$.)

(a) $\sum_{n=0}^{\infty} \frac{n\sqrt{n}}{n^2+3n+6}$

(b) $\sum_{n=2}^{\infty} \log(n^2) - \log(n^2-1)$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{2(n!)^2}$ [Hint: Recall the duplication formula: $\Gamma(2x) = \Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)2^{2x-1}\pi^{-\frac{1}{2}}$.]

4 Show that if a series $\sum_{n=0}^{\infty} a_n$ converges absolutely, then it converges.

5 Find the Fourier series for the following functions: [You may use either the $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ or the $\frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ form for the Fourier series]

(a) $f(x) = x^2 - 2x - \pi^2$ for $-\pi < x \leqslant \pi$, and f 2π-periodic.

(b) $f(x) = x^3 - 3x^2 - \pi^2 x$ for $-\pi < x \leqslant \pi$, and f 2π-periodic.

(c) $f(x) = e^x$ for $-1 \leqslant x < 1$ and f 2-periodic. [Note, the period of this f isn't π.]

6 Find the Fourier sine series for the following functions on the interval $[0,\pi]$.

(a) $f(x) = \cos x$.

(b) $f(x) = 3$.

7 An elastic string of length π, satisfying the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is fixed at one end (so $u(0,t) = 0$) while the other end is made to oscillate so that $u(\pi,t) = \sin t$. Assuming that $u(x,t)$ does separate as a sum of products $\Theta(x)\Phi(t)$ that also satisfy the wave equation, find the motion of the rest of the string. [2 marks]

8 The temperature $u(x,t)$ in a thin metal rod of length π, at position x and time t, satisfies the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, where k is a positive real constant. The rod is heated to a uniform 50°C, then one end is fixed at 0°c, and the other end is fixed at 100°C.

(a) Use separation of variables to find solutions satisfying the boundary conditions $u(0,t) = 0$, $u(\pi,t) = 100$, for all t. [Hint: consider $v(x,t) = u(x,t) - \frac{100}{\pi}x$.]

(b) Use Fourier series to find $u(x,t)$ for $t > 0$. (From the intial condition $u(x,0) = 50$ for all x.)