MATH 3090, Advanced Calculus I

Fall 2006

Mock Final Examination
Model Solutions

1 *Which of the following series of functions converge uniformly on the interval (0,1)? Justify your answers.*

(a) $\sum_{n=1}^{\infty} x^n$

This does not converge uniformly on the interval $(0, 1)$, since $\sum_{n=N}^{\infty} x^n = \frac{x^N}{1-x}$, so if $\epsilon = 1$, given any N, we can choose $x = \left(\frac{1}{2}\right)^{\frac{1}{N}}$, then $\frac{x^N}{1-x} \geqslant \frac{1}{2(1-x)}$, and $1-x \leqslant \frac{1}{2}$, so $\sum_{n=N}^{\infty} x^n \geqslant \epsilon$, so the series does not converge uniformly.

(b) $\sum_{n=1}^{\infty} x^n(1-x)^2$

We can show that $x^n(1-x)^2$ is maximised when $x = \frac{n}{n+2}$ (either by differentiating, or by using the AM-GM inequality). We therefore have $x^n(1-x)^2 \leqslant \frac{n^n}{(n+2)^{n+2}} \leqslant \frac{1}{(n+2)^2}$. Therefore, by the Weierstrass M-test, with $M_n = \frac{1}{(n+2)^2}$, $\sum_{n=1}^{\infty} x^n(1-x)^2$ converges uniformly.

2 *Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence?*

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{n \log n}$

The ratio of consecutive terms is $\frac{(-1)^n x^{2n}(n+1)\log(n+1)}{(-1)^{n+1}x^{2n+2}n\log n} = -\frac{(n+1)\log(n+1)}{x^2 n \log n}$. However, $\frac{(n+1)\log(n+1)}{n \log n} \to 1$ as $n \to \infty$ (it's less than $\frac{(n+1)^2}{n^2}$, which is less than $1 + \frac{3}{n}$) so the ratio of consecutive terms tends to x^2. Therefore, by the ratio test, the series converges whenever $|x| < 1$, and diverges when $|x| > 1$, so the radius of convergence is 1. When $|x| = 1$, the series becomes $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$, which converges by the alternating series test. ($\frac{1}{n \log n}$ is clearly a decreasing function of n, since $n \log n$ is an increasing function of n.)

3 *Which of the following series converge? For series which converge, is the convergence absolute? Justify your answers. (You may assume convergence of geometric series and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$, and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \leqslant 1$.)*

(a) $\sum_{n=0}^{\infty} \frac{n\sqrt{n}}{n^2+3n+6}$

This series diverges by comparison to $\sum_{n=4}^{\infty} \frac{1}{n+2}$ – for $n \geqslant 4$, we have that $n\sqrt{n} \geqslant n+2$, while $n^2+3n+6 \leqslant (n+2)^2$, so for $n \geqslant 4$, $\frac{n\sqrt{n}}{n^2+3n+6} \geqslant \frac{1}{n+2}$.

(b) $\sum_{n=2}^{\infty} \log(n^2) - \log(n^2-1)$

$n^2-1 = (n+1)(n-1)$, so $\log(n^2-1) = \log(n+1) + \log(n-1)$, and so $\log(n^2) - \log(n^2-1) = 2\log n - \log(n+1) - \log(n-1)$. Therefore, $\sum_{n=2}^{N} \log(n^2) - \log(n^2-1) = -\log(N+1) + \log(N) + \log 2 - \log 1$, since the middle terms all cancel. Now, as $N \to \infty$, $\log(N) - \log(N+1) \to 0$, so the series converges (and its limit is $\log 2$). The convergence is absolute, since all terms are positive.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$ *[Hint: Recall the duplication formula:* $\Gamma(2x) = \Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) 2^{2x-1}\pi^{-\frac{1}{2}}$*.]*

$(2n)! = \Gamma(2n+1) = \Gamma\left(2\left(n+\frac{1}{2}\right)\right) = \Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)2^{2n}\pi^{-\frac{1}{2}}$, so the series becomes $\sum_{n=1}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)\pi^{-\frac{1}{2}}}{\Gamma(n+1)}$, and we know that $\frac{\Gamma(x+\alpha)}{\Gamma(x)x^{\alpha}} \to 1$ as $x \to \infty$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$ is approximately $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{\pi}\left(n+\frac{1}{2}\right)^{\frac{1}{2}}}$, which does not converge absolutely. However, given two consecutive terms for n and $n+1$, the ratio of their moduli is $\frac{(2n+1)(2n+2)}{(2n+2)^2}$, which is always less than 1, so the terms of the series are decreasing in modulus, so by the alternating series test, the series converges.

4 *Show that if a series* $\sum_{n=0}^{\infty} a_n$ *converges absolutely, then it converges.*

Given $\epsilon > 0$, there is an N such that $\sum_{n=N}^{\infty} |a_n| < \epsilon$. For, $m_1, m_2 \geqslant N$, we have $\left|\sum_{n=0}^{m_1} a_n - \sum_{n=0}^{m_2}\right| = \left|\sum_{n=m_1+1}^{m_2} a_n\right| \leqslant \sum_{n=m_1+1}^{m_2} |a_n| \leqslant \sum_{n=N}^{\infty} |a_n| < \epsilon$, so the series is Cauchy, and therefore, it converges.

5 *Find the Fourier series for the following functions: [You may use either the* $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ *or the* $\frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx)$ *form for the Fourier series]*

(a) $f(x) = x^2 - 2x - \pi^2$ *for* $-\pi < x \leqslant \pi$*, and* f *2π-periodic.*

$\sum_{n=-\infty}^{\infty} c_n e^{inx}$**:**

The coefficients c_n are given by:

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(x^2 - 2x - \pi^2\right) e^{-inx} dx \\
&= \frac{1}{2\pi}\left(\left[\frac{\left(x^2 - 2x - \pi^2\right) e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (2x-2)\,\frac{e^{-inx}}{-in} dx\right)
\end{aligned}$$

$$= \frac{1}{2\pi}\left(\frac{(-1)^n 4\pi}{-in} - \left[\frac{(2x-2)e^{-inx}}{-n^2}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2\frac{e^{-inx}}{-n^2}dx\right)$$
$$= (-1)^n\frac{2}{-in} + (-1)^n\frac{2}{n^2}$$

for $n \neq 0$. For $n = 0$, $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(x^2 - 2x - \pi^2\right)dx = \pi^2 - \pi^2 = 0$. Therefore, the Fourier series is $f(x) = \sum_{n\neq 0}(-1)^n\left(\frac{2}{-in} + \frac{2}{n^2}\right)e^{inx}$.

$\frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx)$**:**
The coefficients a_n are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2 - 2x - \pi^2\right)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2 - \pi^2\right)\cos(nx)dx$$
$$= \frac{1}{\pi}\left(\left[\frac{(x^2-\pi^2)\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2x\sin(nx)}{n}dx\right)$$
$$= \frac{1}{\pi}\left(\left[\frac{2x\cos(nx)}{n^2}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2\cos(nx)dx\right)$$
$$= (-1)^n\frac{4}{n^2}$$

for $n \neq 0$. As above, $a_0 = 0$.
The b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2 - 2x - \pi^2\right)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\sin(nx)dx$$
$$= (-1)^n\frac{4}{n}$$

So the Fourier series for f is $f(x) = \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos(nx) + (-1)^n\frac{4}{n}\sin(nx)$.

(b) $f(x) = x^3 - 3x^2 - 3\pi^2 x$ for $-\pi < x \leqslant \pi$, and f 2π-periodic.
The derivative $f'(x) = 3x^2 - 6x - 3\pi^2$ is 3 times the function whose Fourier series we computed in (a). The Fourier series in (a) had constant term 0, so we can integrate it termwise, and just worry about the constant term. The constant term of the Fourier series is $\frac{1}{2\pi}\int_{-\pi}^{\pi} 3x^2 dx = \pi^2$ (since the odd terms x^3 and $-\pi^2 x$ cancel when we integrate from $-\pi$ to π). Therefore, the Fourier series is $f(x) = \pi^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}(-1)^n\left(\frac{2}{-n^2} + \frac{2}{-in^3}\right)e^{inx}$,
or $f(x) = \pi^2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}\cos(nx) + (-1)^n\frac{4}{n^3}\sin(nx)$.

(c) $f(x) = e^x$ for $-1 \leqslant x < 1$ and f 2-periodic. [Note, the period of this f isn't π.]

The coefficients c_n are given by $c_n = \frac{1}{2}\int_{-1}^{1} e^{(}1-\pi in)xdx = \frac{1}{2}\left[\frac{e^{(}1-\pi in)x}{1-\pi in}\right]_{-1}^{1} = \frac{(-1)^n}{2}\left(\frac{e-e^{-1}}{1-\pi in}\right)$. Therefore, the Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2}\left(\frac{e-e^{-1}}{1-\pi in}\right)e^{\pi inx}$.

6 *Find the Fourier sine series for the following functions on the interval* $[0,\pi]$.
(a) $f(x) = \cos x$.

The coefficients of the sine series are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)dx = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) + \sin((n-1)x)\,dx$$
$$= \begin{cases} \frac{1}{(n+1)\pi} + \frac{1}{(n-1)\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

So the Fourier sine series is $\cos x = \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{4m}{4m^2-1}\sin(2mx)$ (using the substitution n=2m).

(b) $f(x) = 3$.

The coefficients of the sine series are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} 3\sin(nx)dx = \begin{cases} \frac{6}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Therefore, the Fourier sine series is $3 = \sum_{m=0}^{\infty} \frac{6}{\pi(2m+1)}\sin((2m+1)x)$.

7 *An elastic string of length π, satisfying the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ is fixed at one end (so $u(0,t) = 0$) while the other end is made to oscillate so that $u(\pi,t) = \sin t$. Assuming that $u(x,t)$ does separate as a sum of products $\Theta(x)\Phi(t)$ that also satisfy the wave equation, find the motion of the rest of the string. [2 marks]*

We use separation of variables on the wave equation: if u is of the form $u(x,t) = \Theta(x)\Phi(t)$, then Θ and Φ satisfy $\Theta(x)\ddot{\Phi}(t) = c^2\Theta''(x)\Phi(t)$, and therefore, $\frac{\ddot{\Phi}(t)}{\Phi(t)} = c^2\frac{\Theta''(x)}{\Theta(x)}$. But the right-hand side of this equation depends only on x, while the left-hand side depends only on t, so they must both be constant. Also, to get the boundary condition $u(\pi,t) = \sin t$, we must have $\Phi(t) = \lambda\sin(t)$ for a constant $\lambda = \Theta(\pi)$. This means that $\frac{\ddot{\Phi}(t)}{\Phi(t)} = -1$, and therefore, $\frac{\Theta''(x)}{\Theta(x)} = \frac{-1}{c^2}$, And so, $\Theta(x) = a\sin\left(\frac{x}{c}\right) + b\cos\left(\frac{x}{c}\right)$. However, from the condition $u(0,t) = 0$ for all t, we must have that $b = 0$, so the solution must be $u(x,t) = \frac{\sin t \sin\left(\frac{x}{c}\right)}{\sin\left(\frac{\pi}{c}\right)}$.

8 The temperature $u(x,t)$ in a thin metal rod of length π, at position x and time t, satisfies the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, where k is a positive real constant. The rod is heated to a uniform $50°$C, then one end is fixed at $0°$c, and the other end is fixed at $100°$C.

(a) Use separation of variables to find solutions satisfying the boundary conditions $u(0,t) = 0$, $u(\pi,t) = 100$, for all t. [Hint: consider $v(x,t) = u(x,t) - \frac{100}{\pi}x$.]

$v(x,t) = u(x,t) - \frac{100}{\pi}x$ satisfies $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = k\frac{\partial^2 v}{\partial x^2}$, and has boundary conditions $v(0,t) = v(\pi,t) = 0$. If we suppose $v(x,t) = \Theta(x)\Phi(t)$, then Θ and Φ satisfy: $\Theta(x)\dot{\Phi}(t) = k\Theta''(x)\Phi(t)$, and therefore, $\frac{\dot{\Phi}(t)}{\Phi(t)} = k\frac{\Theta''(x)}{\Theta(x)}$. The left-hand side depends only on t, while the right-hand side depends only on x, so they must both equal some constant λ. The boundary conditions $\Theta(0) = \Theta(\pi) = 0$ mean that we must have $\Theta(x) = a\sin(nx)$ for some integer n, and some a. This means that $\lambda = n^2$, so $\Phi(t) = be^{-kn^2 t}$ for some b. Therefore, the solution to the equation with the boundary conditions is $u(x,t) = \frac{100}{\pi}x + \sum_{n=1}^{\infty} b_n e^{-kn^2 t}\sin(nx)$ for some values of b_n.

(b) Use Fourier series to find $u(x,t)$ for $t > 0$. (From the intial condition $u(x,0) = 50$ for all x.)

We know that $u(x,0) = 50$ for all x, so $v(x,0) = 50 - \frac{100}{\pi}x$. We can therefore use Fourier series to find the coefficients b_n above, by substituting $t = 0$. We get

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}\left(50 - \frac{100}{\pi}x\right)\sin(nx)dx \\ &= \frac{100}{\pi}\left(\left[-\frac{\left(1-\frac{2}{\pi}x\right)\cos(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{2}{\pi}\cos(nx)dx\right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{200}{\pi n} & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Therefore, we have that $u(x,t) = \frac{100}{\pi}x + \sum_{m=1}^{\infty}\frac{100}{\pi m}e^{-4km^2 t}\sin(2mx)$.