MATH 3090, Advanced Calculus I
Fall 2006
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Mock Midterm Examination
Model Solutions

1 *Show that if the series $\sum_{n=0}^{\infty} a_n$ converges, where $a_n \geqslant 0$ for all n, then so does $\sum_{n=0}^{\infty} a_n^2$.*

Since $\sum_{n=0}^{\infty} a_n$ converges, we must have $a_n \to 0$ as $n \to \infty$. Therefore, we can choose an N such that $(\forall n \geqslant N)(a_n < 1)$. Now for $n \geqslant N$, $a_n^2 < a_n$, so $\sum_{n=N}^{\infty} a_n^2$ converges by comparison to $\sum_{n=0}^{\infty} a_n$.

2 *Which of the following series of functions converge uniformly on the interval (0,1)? Justify your answers.*

(a) $\sum_{n=1}^{\infty} f_n(x)$ *where* $f_n(x) = \begin{cases} \frac{1}{n^2} & \text{if } x < \frac{1}{n} \\ 0 & \text{if } x \geqslant \frac{1}{n} \end{cases}$

For every n, and every x, $f_n(x) \leqslant \frac{1}{n^2}$, so $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly by the Weierstrass M-test with $M_n = \frac{1}{n^2}$.

(b) $\sum_{n=1}^{\infty} \frac{x+\frac{1}{n^2}}{(1+x)^{n^2}}$ *[Hint: substitute $y = n^2 x$, and expand the denominator]*

If we let $y = n^2 x$, then the fraction is $\frac{\frac{y+1}{n^2}}{\left(1+\frac{y}{n^2}\right)^{n^2}}$. However,

$$\left(1+\frac{y}{n^2}\right)^{n^2} = 1 + n^2\frac{y}{n^2} + \begin{pmatrix} n^2 \\ 2 \end{pmatrix}\left(\frac{y}{n^2}\right)^2 + \ldots \geqslant 1 + y$$

so $\frac{\frac{y+1}{n^2}}{\left(1+\frac{y}{n^2}\right)^{n^2}} \leqslant \frac{1}{n^2}$. Therefore, the series converges uniformly by the Weierstrass M-test with $M_n = \frac{1}{n^2}$.

3 *Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence?*

(a) $\sum_{n=0}^{\infty} \frac{x^n}{3^{2n+1}}$

The ratio of consecutive terms is $\frac{x^{n+1}3^{2n+1}}{x^n 3^{2n+3}} = \frac{x}{9}$. Therefore, for $|x| < 9$, the series converges by the ratio test, while for $|x| > 9$, it diverges by the ratio test. Thus, the radius of convergence is 9. When $|x| = 9$, the series diverges, because all the terms have modulus $\frac{1}{3}$, so they do not converge to 0.

(b) $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$ *(You may assume that* $\left(1+\frac{1}{n}\right)^n \to e$ *as* $n \to \infty$*.)*

The ratio of consecutive terms in this series is $\frac{(n+1)^{n+1}x^{n+1}n!}{(n+1)!n^n x^n} = \frac{(n+1)^{n+1}x}{(n+1)n^n} = \left(1+\frac{1}{n}\right)^n x$. As $n \to \infty$, this converges to ex, so the series converges if $ex < 1$, and diverges if $ex > 1$. The radius of convergence is therefore $\frac{1}{e}$.

I should have been more careful stating this question – it wasn't reasonable to expect you to determine whether the series converges for $x = \pm\frac{1}{e}$. We will see shortly when we cover Stirling's formula that the series diverges for $x = \frac{1}{e}$ and converges when $x = -\frac{1}{e}$ by the alternating series test. (In fact we can show the latter by observing that $\left(1+\frac{1}{n}\right)^n < e$ for all n – when we expand the bracket the terms we get are all less than the corresponding terms in the Taylor series for e.)

4 State and prove the Bolzano-Weierstrass theorem.

Bolzano-Weierstrass Theorem: Any bounded sequence of real numbers has a convergent subsequence.

Proof: Let a_n be a sequence of real numbers. We will show that it has a monotone subsequence, then if a_n is bounded, this subsequence will be convergent by the monotone convergence axiom.

To show that a_n has a monotone subsequence, we will call a natural number n *far-seeing* if $(\forall m > n)(a_n \geqslant a_m)$. Now either there are infinitely many far-seeing n: in which case, the subsequence a_{n_i}, where the n_i are the far-seeing n, is a decreasing subsequence; or there are only finitely many far-seeing n, in which case, there is a largest far-seeing N, so for any $n > N$, there is an $m > n$ with $a_n < a_m$. We can therefore choose $n_1 = N+1$, and inductively define n_{i+1} to be the first number larger than n_i to satisfy $a_{n_i} < a_{n_{i+1}}$. The a_{n_i} then form an increasing subsequence of the a_n. Therefore, in either case, we have a monotone subsequence.

5 Which of the following series converge? For series which converge, is the convergence absolute? Justify your answers. (You may assume convergence of geometric series and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *for* $p > 1$*, and divergence of* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *for* $p \leqslant 1$*.)*

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2}{3n^2+4n+5}$

As $n \to \infty$, $\frac{n^2+2}{3n^2+4n+5} \to \frac{1}{3}$, so the terms do not converge to zero. Therefore, the sum cannot converge.

(b) $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}-\arctan n}{n}$

solution 1(integral test): $\frac{\frac{\pi}{2}-\arctan x}{x}$ is a decreasing function of x for $x > 0$, and it converges to 0 as $x \to \infty$. Therefore, we can apply the integral test, to say that $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}-\arctan n}{n}$ converges if and only if $\int_1^{\infty} \frac{\frac{\pi}{2}-\arctan x}{x} dx$ converges. We perform the substitution $x = \tan\theta$, to get: $\int_{\arctan 1}^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-\theta}{\tan\theta\cos^2\theta} d\theta$ But $\tan\theta\cos^2\theta = \sin\theta\cos\theta$, and $\frac{\frac{\pi}{2}-\theta}{\cos\theta}$ converges to 1 as $\theta \to \frac{\pi}{2}$. Therefore, the integral is bounded, so it converges. Therefore, $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}-\arctan n}{n}$ converges.

solution 2: $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$, so $\frac{\pi}{2} - \arctan n = \arctan\left(\frac{1}{n}\right)$, so the sum is $\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{n}\right)}{n}$, and for $n \geqslant 1$, $\arctan\left(\frac{1}{n}\right) \leqslant \frac{1}{n}$. Thus, $\frac{\arctan\left(\frac{1}{n}\right)}{n} \leqslant \frac{1}{n^2}$, so the series converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

6 *Show that if $f_n \to f$ uniformly on the interval $[a, b]$, and all the f_n are continuous on $[a, b]$, then f is continuous on $[a, b]$. If the f_n are all differentiable at some $x \in [a, b]$, must f be differentiable at x? Give a proof or a counterexample.*

We need to show that given $\epsilon > 0$, and $x \in [a, b]$, there is a $\delta > 0$, such that $(\forall y \in [a, b])(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$. Since the $f_n \to f$ uniformly on $[a, b]$, we can find an N such that $(\forall x \in [a, b])(|f_N(x) - f(x)| < \frac{\epsilon}{3})$ (indeed we can find N such that this holds for every $n \geqslant N$). Now, f_N is continuous, so we can choose a $\delta > 0$, such that

$$(\forall y \in [a, b])(|y - x| < \delta \Rightarrow |f_N(y) - f_N(x)| < \frac{\epsilon}{3})$$

Now if $|x - y| < \delta$ then

$$\begin{aligned}&|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \leqslant \\ &|f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

which is what we needed for f to be continuous.

f does not have to be differentiable at $x \in [a, b]$ even if all the f_n are differentiable there. For example, if $a = -1$, $b = 1$ and $f_n(x) = \begin{cases} -x & \text{if } x < -\frac{1}{n} \\ \frac{1}{n} & \text{if } -\frac{1}{n} \leqslant x \leqslant \frac{1}{n} \\ x & \text{if } x > \frac{1}{n} \end{cases}$ then f_n converges uniformly to $f(x) = |x|$, but every f_n is differentiable at 0, while f is not differentiable at 0.