MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 6
Model Solutions

1 *Show that* $\cos n\theta = \sum_{j=0}^{\frac{n}{2}}(-1)^j \begin{pmatrix} n \\ 2j \end{pmatrix} \cos^{n-2j}\theta \sin^{2j}\theta$ *and that* $\sin n\theta = \sum_{j=0}^{\frac{n}{2}}(-1)^j \begin{pmatrix} n \\ 2j+1 \end{pmatrix} \cos^{n-2j-1}\theta \sin^{2j+1}\theta$. *[Hint: use* $e^{i\theta} = \cos\theta + i\sin\theta$ *and the binomial formula.]*

$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$. By the binomial theorem:

$$(\cos\theta + i\sin\theta)^n = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} i^j \cos^{n-j}\theta \sin^j\theta$$

We know that $i^{4j} = 1$, $i^{4j+1} = i$, $i^{4j+2} = -1$ and $i^{4j+3} = -i$, so, taking real and imaginary parts, we get:

$$\cos n\theta = \sum_{j=0}^{\frac{n}{2}}(-1)^j \begin{pmatrix} n \\ 2j \end{pmatrix} \cos^{n-2j}\theta \sin^{2j}\theta$$

and

$$\sin n\theta = \sum_{j=0}^{\frac{n}{2}}(-1)^j \begin{pmatrix} n \\ 2j+1 \end{pmatrix} \cos^{n-2j-1}\theta \sin^{2j+1}\theta$$

2 *Which non-zero complex numbers* z *have the property that* $z + \frac{1}{z}$ *is real?*

Let $z = a + ib$. Then $\frac{1}{z} = \frac{a-ib}{a^2+b^2}$, so the imaginary part of $z + \frac{1}{z}$ is $b\left(1 - \frac{1}{a^2+b^2}\right)$. $z + \frac{1}{z}$ is real when this is zero, which happens either when $b = 0$, or when $a^2 + b^2 = 1$. Therefore, the complex numbers that have the property given are the real numbers, and the numbers with modulus 1.

3 *Evaluate the following improper integrals*
(a) $\int_0^\infty \int_t^\infty e^{-x^2} dx dt$

The integral converges absolutely, so we can change the order of integration to get $\int_0^\infty \int_0^x dt e^{-x^2} dx = \int_0^\infty xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$.

(b) $\int_0^\infty \frac{1-\cos t}{t^2}dt$. *[Hint: you can calculate* $\int_0^\infty \frac{\sin xt}{t}dt$ *for* $x > 0$ *by the change of variable* $u = xt$*. Now integrate with respect to* x*.]*

There was a mistake in setting this question. The question I intended to set was:

(b) $\int_0^\infty \frac{\cos at-\cos bt}{t^2}dt$ *for* $0 < a < b$*. [Hint: you can calculate* $\int_0^\infty \frac{\sin xt}{t}dt$ *for* $x > 0$ *by the change of variable* $u = xt$*. Now integrate with respect to* x*.]*

The question I actually set is the $a = 0, b = 1$ case of this question. The solution to the question I intended to set is:

Applying the change of variable $u = xt$, we get $\int_0^\infty \frac{\sin xt}{t}dt = \int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}$ for all $x > 0$. The change of variable $u = xt$ also makes it clear that the convergence of the integral is uniform in x, for $x \geqslant a$ as for larger values of x, the integral converges more quickly. Therefore, we can change the order of integration to get

$$\int_0^\infty \frac{\cos at - \cos bt}{t^2}dt = \int_0^\infty \int_a^b \frac{\sin xt}{t}dxdt = \int_a^b \int_0^\infty \frac{\sin xt}{t}dtdx = \int_a^b \frac{\pi}{2}dx$$
$$= \frac{(b-a)\pi}{2}$$

However, the convergence is not uniform as $a \to 0$, so this won't give the solution to the problem that I actually set. In fact, because the error term $\int_0^\infty \frac{1-\cos at}{t^2}dt$ is bounded by $\int_0^R \frac{a}{2}dt + \int_R^\infty \frac{2}{t^2}$, we can show that it does tend to 0 as $a \to 0^+$, and therefore, deduce that $\int_0^\infty \frac{1-\cos t}{t^2}dt = \frac{\pi}{2}$, but this was not something I expected you to notice.

4 *Do the following series converge? Justify your answers.*
(a) $\sum_{n=1}^\infty \frac{1\times 5\times 9\times\cdots\times(4n+1)}{3\times 7\times 11\times\cdots\times(4n+3)}$

The numerator of the fraction is $\frac{4^{n+1}\Gamma\left(n+\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$, while the denominator is $\frac{4^{n+1}\Gamma\left(n+\frac{7}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$. Therefore, the series is

$$\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sum_{n=0}^\infty \frac{\Gamma\left(n+\frac{5}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)}$$

As $n \to \infty$, the ratio of gamma functions is approximately $n^{-\frac{1}{2}}$ (By Theorem 7.57). Therefore, the series diverges by comparison to $\sum_{n=1}^\infty n^{-\frac{1}{2}}$.

(b) $\sum_{n=1}^{\infty} \frac{(2n)!^4}{(4n)!(n!)^4}$

Using Stirling's formula, the fractions being summed are approximately

$$\frac{\left((2n)^{2n+\frac{1}{2}}\right)^4 e^{8n}}{(4n)^{4n+\frac{1}{2}}\left(n^{n+\frac{1}{2}}\right)^4 e^{8n}}$$

This cancels to $\frac{4}{n^{\frac{1}{2}}}$, so the series diverges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$.