All countable monoids embed into the monoid of the infinite random graph

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Dedicated to Jarik Nešetřil on his 60th birthday.

Abstract

We prove that the full transformation monoid on a countably infinite set is isomorphic to a submonoid of $\mathrm{End}(R)$, the endomorphism monoid of the infinite random graph R. Consequently, $\mathrm{End}(R)$ embeds each countable monoid, satisfies no nontrivial monoid identity, and has an undecidable universal theory.

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The infinite random graph, written R, has many remarkable properties which have attracted the attention of several researchers, including graph theorists, logicians, and algebraists. The graph R is the unique (up to isomorphism) countable graph that satisfies the *existentially closed* adjacency property: for all finite disjoint sets of vertices A and B, there is a vertex joined to each vertex of A and not joined nor equal to a vertex of B. An interesting property of R, known to Fraïssé [6] in 1953, is that it is a *universal* graph: each countable graph is isomorphic to an induced subgraph of R. For other properties of R, the reader is directed to the surveys of Cameron [3,4]. All graphs we consider are undirected, countable, and simple. Let $\aleph_0$ denote the cardinality of $\mathbb{N}$. For additional background on graphs and graph homomorphisms, the reader is directed to the new excellent text of Hell and Nešetřil [7].

While the automorphism group of R has been thoroughly investigated (see the references in [3,4]), properties of the endomorphism monoid of R have been largely overlooked. In [2], the first two authors studied the monoid $\mathrm{End}(R)$, and characterized properties of its retracts. This monoid was further studied in [1,5].

We prove in this short note that $\mathrm{End}(R)$ is universal as a monoid; that is, it contains as a submonoid an isomorphic copy of each countable monoid. To this end, we use a well-known fact that each countable monoid embeds in the *full transformation monoid $T(X)$*, the monoid of all mappings from X to itself under composition, where X is a countably infinite set. Our main result is now as follows.

Theorem 1 *If X is a countable set, then $T(X)$ embeds in $\mathrm{End}(R)$.*

The principal idea is to use the universality property of R. We start with a graph G, and then inductively construct a graph R_G containing it, so that $R_G \cong R$. More formally, let G be a fixed countable graph. First, we define G^* by adding a new vertex v_S for each finite subset $S \subseteq V$, so that v_S is joined to the vertices belonging to S and no other vertex from G^*. Define a chain of graphs by setting $G_0 = G$ and $G_{n+1} = G_n^*$ for all $n \geqslant 0$. The union of this chain is denoted by R_G; that is,

$$R_G = \bigcup_{n \in \mathbb{N}} G_n.$$

The above is one of the canonical ways of constructing R (see for example [3]), as it is quickly verified that R_G satisfies the existentially closed property. We record these facts in the form of the following lemma.

Lemma 1 *For any countable graph G, the graph R_G is isomorphic to R.*

The key ingredient in our main proof is the fact that any endomorphism of G can be extended to a endomorphism of G^*.

Lemma 2 *If f is an endomorphism of a graph G, then there is an endomorphism f^* of G^* that extends f.*

PROOF. If $G = (V, E)$ and $G^* = (V^*, E^*)$, then define $f : V^* \to V^*$ in the following way:

$$
f^*(x) = \begin{cases} f(x) \text{ if } x \in V, \\ \\ v_{f(S)} \text{ if } x = v_S \text{ for a finite } S \subseteq V. \end{cases}
$$

Clearly, $f^* \upharpoonright V = f$. To see that f^* is a graph homomorphism, since any two vertices $x, y \in V^* \setminus V$ are not joined, we need only to consider the case when $x \in V$ and $y \in V^* \setminus V$. Then $y = v_S$ for a finite $S \subseteq V$ and $x \in S$. However, these assumptions imply $f(x) \in f(S)$ and $f(y) = v_{f(S)}$, showing that $f(x)$ and $f(y)$ are joined, as required. $\qquad\square$

Now we exploit the feature of the infinite co-clique $\overline{K_{\aleph_0}}$ that each transformation of its vertex set is a graph endomorphism.

PROOF OF THEOREM 1. Start with $\overline{K_{\aleph_0}}$ as G_0. As already noted, $\text{End}(G_0) \cong T(X)$ (in fact, we may take the vertex set of G_0 as X). By Lemma 1, $R_{G_0} \cong R$, so we identify R with R_{G_0} in the sequel.

By Lemma 2 and a straightforward induction, it follows that for any $f \in T(X)$ and for each $n > 0$ there is an endomorphism f_n of the graph G_n such that we have $f_n \upharpoonright V(G_0) = f$ and, moreover, $f_n \upharpoonright V(G_m) = f_m$ for all $0 < m < n$. Define a mapping $\phi : T(X) \to \text{End}(R_{G_0})$ by

$$
\phi(f) = \bigcup_{n \in \mathbb{N}} f_n,
$$

where $f_0 = f$. This mapping is well-defined, since for any $x, y \in V(R_{G_0})$ there is an $n \geqslant 0$ such that $x, y \in V(G_n)$. Thus, if x and y are joined by an edge, so are $(\phi(f))(x) = f_n(x)$ and $(\phi(f))(y) = f_n(y)$, as f_n is an endomorphism of G_n.

It is straightforward to see that ϕ is injective, as $\phi(f) \upharpoonright V(G_0) = f$. Hence, it remains to show that ϕ is a homomorphism of monoids, that is, that

$$
\phi(fg) = \phi(f)\phi(g)
$$

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holds for all $f, g \in T(X)$. We immediately have that ϕ preserves the identity transformation. It is sufficient to prove that $(fg)_n = f_n g_n$ for all $n \geqslant 0$, provided that we have $f_{n+1} = f_n^*$ and $g_{n+1} = g_n^*$ as in the proof of Lemma 2. The required equality is clear for $n = 0$. For the case of $n + 1$, we must show that

$$(fg)_{n+1}(x) = f_{n+1} g_{n+1}(x)$$

for all $x \in V(G_{n+1})$. Bearing in mind the induction hypothesis, we may assume that $x \in V(G_{n+1}) \setminus V(G_n)$. Therefore, $x = v_S$ for a unique finite $S \subseteq V(G_n)$, implying

$$(fg)_{n+1}(x) = v_{(fg)_n(S)} = v_{f_n(g_n(S))} = f_{n+1}(v_{g_n(S)}) = f_{n+1}(g_{n+1}(x)),$$

where the second equality follows by the induction hypothesis. $\qquad\square$

Theorem 1 has the following consequences. We refer the reader to Hodges [8] for any terms not explicitly defined.

Corollary 2 *The monoid* $\mathrm{End}(R)$ *does not satisfy any non-trivial monoid identity. In particular,* $\mathrm{End}(R)$ *generates the variety of all monoids.*

PROOF. Since every countable monoid embeds into $\mathrm{End}(R)$ by Theorem 1, so does the free monoid on a countable set of generators, written $F(X)$. If there were an equation $s = t$ in the language of monoids that is not consequence of the associative law, and satisfied by $\mathrm{End}(R)$, then $s = t$ would be satisfied by $F(X)$, which is a contradiction. $\qquad\square$

Corollary 3 *The universal theory of* $\mathrm{End}(R)$ *is undecidable.*

PROOF. We first observe that the universal theory of $\mathrm{End}(R)$ equals the universal theory of all monoids. To see this, note that since every countable monoid embeds into $\mathrm{End}(R)$ by Theorem 1, every universal sentence true in $\mathrm{End}(R)$ will be true in all countable monoids and, by the Löwenheim-Skolem Theorem (see [8]), in all monoids.

It is well-known that the universal theory of monoids (semigroups) is undecidable. This fact follows this by the existence of a semigroup with an undecidable word problem [10,11]. Hence, the universal theory of $\mathrm{End}(R)$ is undecidable. $\square$

We note that the monoid $\mathrm{End}(R)$ (which has cardinality $2^{\aleph_0}$) does not embed all monoids of cardinality at most $2^{\aleph_0}$. The reason for this is that $T(X)$, where X is countable, does not embed all monoids of cardinality at most $2^{\aleph_0}$, and

by Theorem 1, $T(X)$ and $\mathrm{End}(R)$ are mutually embeddable. For example, an uncountable direct sum of countable simple groups does not embed into $T(X)$ [9]. We do not know, however, exactly which uncountable monoids embed in $\mathrm{End}(R)$.

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