THE SEARCH FOR N-E.C. GRAPHS

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ABSTRACT. Almost all finite graphs have the n-e.c. adjacency property, although until recently few explicit examples of such graphs were known. We survey some recently discovered families of explicit finite n-e.c. graphs, and present a new construction of strongly regular n-e.c. arising from affine planes.

1. INTRODUCTION

Adjacency properties of graphs were first discovered and investigated by Erdős, Rényi [22] in their generative work on random graphs. An *adjacency property* is a global property of a graph asserting that for every set S of vertices of some fixed type, there is a vertex joined to some of the vertices of S in a prescribed way. The so-called n-e.c. adjacency property has received much recent attention, and is the focus of this survey. For a positive integer n, a graph is *n-existentially closed* or *n-e.c.*, if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B can be empty), there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. We say that z is *correctly joined* to A and B. Hence, for all n-subsets S of vertices, there exist 2^n vertices joined to S in all possible ways.

For example, a 1-e.c. graph is one with neither isolated nor universal vertices. A graph is 2-e.c. if for each pair of distinct vertices a and b, there are four vertices not equalling a and b joined to them in all possible ways. See Figure 2 for an example of a 2-e.c. graph.

Although the n-e.c. property is straightforward to define, it is not obvious from the definition that graphs with the property exist. However, as first proved in [22], almost all finite graphs are n-e.c. We sketch a proof here for completeness. For a positive integer m, the probability

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space $G(m, \frac{1}{2})$ consists of graphs with vertices $\{0, \dots, m-1\}$ so that two distinct vertices are joined independently and with probability $\frac{1}{2}$.

Theorem 1. *Fix an integer $n > 1$. With probability 1 as $m \to \infty$, $G(m, \frac{1}{2})$ satisfies the n-e.c. property.*

Proof. Fix an n-set X of vertices, and fix subsets A and B of X with $A \cup B = X$. For a given $z \notin X$, the probability z is not correctly joined to A and B is

$$1 - \frac{1}{2}^n.$$

The probability that no node of G is correctly joined to A and B is therefore

$$\left(1 - \frac{1}{2}^n\right)^{m-n}.$$

As there are $\binom{m}{n}$ choices for X and 2^n choices of A and B in X, the probability that $G(m, 12)$ is not n-e.c. is at most

$$\binom{m}{n} 2^n \left(1 - \frac{1}{2}^n\right)^{m-n}$$

which tends to 0 as $m \to \infty$. □

Theorem 1 implies that there are many examples of n-e.c. graphs (note also that it easily generalizes by replacing $\frac{1}{2}$ with any fixed real number $p \in (0, 1)$). Despite this fact, until recently only one explicit family of n-e.c. graphs was known: the Paley graphs (see Section 3). This paradoxical quality of n-e.c. graphs being both common and rare has intrigued many researchers with differing backgrounds such as graph theorists, logicians, design theorists, probabilists, and geometers.

If a graph is n-e.c. for all n, then the graph is called *e.c.* (note that any e.c. graph is infinite). Any two countable e.c. graphs are isomorphic; the isomorphic type is named the *infinite random* or *Rado* graph, and is written R. An important result of Erdős and Rényi [22] states that with probability 1, a countably infinite random graph is isomorphic to R. The (deterministic) graph R has been the focus of much research activity. See [15] for a survey on R.

With the example of R in mind, if a finite graph G is n-e.c., then G may be viewed as a finitary version of R. Hence, the n-e.c. properties are one deterministic measure of randomness in a graph. Other notions of randomness in graphs were proposed and thoroughly investigated. Two such notions (which we will not discuss here) are *quasi-randomness* [20] and *pseudo-randomness* [37]. Many of the graphs in this survey - such as Paley graphs - satisfy these properties. However, these randomness properties do not necessarily imply the n-e.c. properties. Examples are given in [17] which are pseudo-random but not 4-e.c. For additional information on pseudo-randomness and similar properties in graphs, the interested reader is directed to the surveys [19] and [32].

In the last few years, many new explicit families of finite n-e.c. graphs were discovered. Our goal in this survey is to summarize some of the recent constructions of explicit n-e.c. graphs. The techniques used in these constructions are diverse, emanating from probability theory and random graphs, finite geometry, number theory, design theory, and matrix theory. This diversity makes the topic of n-e.c. graphs both challenging and rewarding. More constructions of n-e.c. graphs likely remain undiscovered, and it is our desire that this survey will help foster work in this area.

Apart from their theoretical interest, adjacency properties have recently emerged as an important tool in research on real-world networks. Several evolutionary random models for the evolution of the web graph and other self-organizing networks have been proposed. The n-e.c. property and its variants have been used in [12] and [31] to analyze the graphs generated by the models, and to help find distinguishing properties of the models.

After an introduction to n-e.c. graphs and a summary of background material in Section 2, we will focus on three categories of constructions: Paley graphs and their variants (Section 3), graphs defined using combinatorial designs and matrices (Section 4), and graphs arising from finite geometry (Section 5). In Section 5, we construct a new family of explicit strongly regular n-e.c. graphs derived from affine planes; the proof of the n-e.c. property for this family is elementary.

As a disclaimer, while we present a thorough overview of several recent constructions of n-e.c. graphs, we cannot guarantee that this survey is exhaustive. A useful reference for background topics from graph theory, design theory, and geometry is [14]. *All graphs considered are simple, undirected, and finite* unless otherwise stated. All logarithms are in base e.

2. Background on N-e.c. graphs

While adjacency properties similar to n-e.c. were first studied by Erdős and Rényi, the notion of an *existentially closed graph* (or more generally, a first-order existentially closed structure) was introduced by the logician Abraham Robinson in the 1960s. Existentially closure may be thought of as a generalization of algebraic closure in field theory. (The moniker *n-e.c. structure* was first explicitly used in the author's

Ph.D. thesis [7]; see also [8]). While a formal definition of e.c. graph is outside the scope of this survey, an important fact is that a graph is e.c. if and only if it is n-e.c. for all n. The n-e.c. properties and related adjacency properties also play a role in finite model theory, and were used in [23, 25] to prove the zero-one law in the first-order theory of graphs. For more on existentially closed structures and zero-one laws, see [28].

If a graph G has the n-e.c. property, then G possesses other structural properties summarized in the following theorem (whose proof is straightforward, and so is omitted). The complement of a graph G is denoted by $\overline{G}$, and the chromatic and clique numbers of G are denoted by $\chi(G)$, $\omega(G)$, respectively. Given a vertex x, the induced subgraph formed by deleting x is denoted by $G - x$, and $N(x) = \{y \in V(G) : xy \in E(G)\}$, $N^c(x) = \{y \in V(G) : x \neq y \text{ and } xy \in E(G)\}$. If S is a set of vertices in G, then we write $G \restriction S$ for the subgraph induced by S in G.

Theorem 2. *Fix n a positive integer, and let G be an n-e.c. graph.*

(1) *The graph G is m-e.c., for all $1 \leq m \leq n - 1$.*
(2) *The graph G has order at least $n + 2^n$, and has at least $n2^{n-1}$ many edges.*
(3) *The graph $\overline{G}$ is n-e.c.*
(4) *Each graph of order at most $n + 1$ embeds in G. In particular, $\chi(G), \omega(G) \geq n + 1$.*
(5) *If $n > 1$, then for each vertex x of G, each of the graphs*
$$G - x, G \restriction N(x), \text{ and } G \restriction N^c(x)$$
are $(n - 1)$-e.c.

For each positive integer n, define $m_{ec}(n)$ to be the minimum order of an n-e.c. graph. It is straightforward to see that $m_{ec}(1) = 4$. There are exactly three non-isomorphic 1-e.c. graphs of order 4: $2K_2, C_4$, and P_4; see Figure 1. By Theorem 2 (5), $m_{ec}(2) \geq 9$. For two graphs G

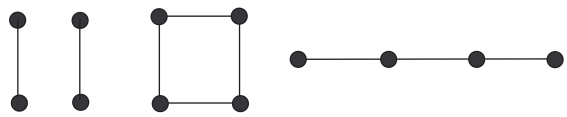

FIGURE 1. The 1-e.c. graphs of minimum order.

and H, the *Cartesian product* of G and H, written $G \square H$, has vertices $V(G) \times V(H)$ and edges $(a, b)(c, d) \in E(G \square H)$ if and only if $ac \in E(G)$

and $b = d$, or $a = c$ and $bd \in E(H)$. The graph $K_3 \square K_3$ is shown in Figure 2. In [13] it was first noted that $m_{ec}(2) = 9$ and $K_3 \square K_3$ is 2-e.c.

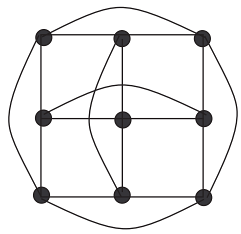

FIGURE 2. The unique 2-e.c. graph of minimum order.

In [9] it was observed that $K_3 \square K_3$ is in fact the unique isomorphism type of 2-e.c. graph of order 9.

By Theorem 2 (5), $m_{ec}(3) \geq 19$. However, a 19-vertex 3 -e.c. graph would be 9-regular, and so $m_{ec}(3) \geq 20$. A computer search of [34] showed that in fact, $m_{ec}(3) \geq 22$. A computer search of [9] found two non-isomorphic 3-e.c. graphs of order 28 (although others may exist). Hence, $22 \leq m_{ec}(3) \leq 28$. The exact determination of $m_{ec}(n)$ where $n \geq 3$ is a difficult open problem.

For $n > 1$, by Theorem 2 (5) we have that $m_{ec}(n) \geq 2m_{ec}(n-1)+1$. In particular, $m_{ec}(n) = \Omega(2^n)$. The proof of Theorem 1 shows that $m_{ec}(n) = O(2^n n^2)$. Hence,

$$\lim_{n\to\infty} m_{ec}(n)^{1/n} = 2.$$

Another open problem surrounding the function $m_{ec}(n)$ is to determine whether

$$\lim_{n\to\infty} \frac{m_{ec}(n)}{2^n}$$

exists, and if so, to find its value.

3. Paley graphs and their variants

Most of the known explicit n-e.c. graphs are strongly regular. A k -regular graph G with v vertices, so that each pair of joined vertices has exactly λ common neighbours, and each pair of non-joined vertices has exactly μ common neighbours is called a *strongly regular graph*; we say that G is a $\mathrm{SRG}(v, k, \lambda, \mu)$.

The first family of explicit graphs that were discovered to contain n -e.c. members for all n were Paley graphs. Paley graphs are defined over certain finite fields, and it has long been observed that they satisfy many of the properties of the random graph $G(n, \frac{1}{2})$ (for instance, they are quasi-random in the sense of [20]). The *Paley graph* of order q, for a prime power $q \equiv 1 \pmod 4$, is the graph P_q whose vertices are the elements of the finite field $\mathrm{GF}(q)$ in which two distinct vertices x and y are joined if and only if $x - y$ is a square in $\mathrm{GF}(q)$. For example, P_5 is isomorphic to the 5 -cycle, while P_9 is isomorphic to the graph $K_3 \square K_3$ of Figure 2.

Paley graphs have many properties as described in the following theorem, proved using properties of finite fields.

Theorem 3. *Fix q a prime power with $q \equiv 1 \pmod 4$.*

(1) The graph P_q is a $\mathrm{SRG}(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$.
(2) The graph P_q is *self-complementary*; that is $P_q \cong \overline{P_q}$.
(3) The graph P_q is *symmetric*; that is, it is vertex- and edge-transitive.

As proved in [4, 6], sufficiently large Paley graphs are n-e.c.

Theorem 4. *If $q > n^2 2^{2n-2}$, then P_q is n-e.c.*

The proofs of Theorem 4 in [4, 6] each use a famous result from number theory on character sum estimates, namely Weil's proof of the Riemann hypothesis over finite fields. For a proof of the following theorem and additional background, see [36].

Theorem 5. *Let χ be a nontrivial character of order d over $\mathrm{GF}(q)$. Suppose that $f(x)$ is a polynomial over $\mathrm{GF}(q)$ with exactly m distinct zeros and is not of the form $c(g(x))^d$, where $c \in \mathrm{GF}(q)$ and $g(x)$ is a polynomial over $\mathrm{GF}(q)$. Then*

$$\left| \sum_{x \in \mathrm{GF}(q)} \chi(f(x)) \right| \leq (m-1)q^{1/2}.$$

Proofs of Theorem 4 using Theorem 5 are similar to those given by Graham and Spencer [26] for tournaments. If we consider $q \equiv 3 \pmod 4$ in the definition of Paley graphs, then we obtain a tournament called the *Paley tournament* $\overrightarrow{P_q}$. An n-e.c. tournament is defined in an analogous way to an n-e.c. graph. A proof that sufficiently large Paley tournaments are n-e.c. was given in [26]. See Figure 3 for the 2-e.c. tournament $\overrightarrow{P_7}$ (which is the unique isomorphism type of 2-e.c. tournament of order 7; see [10]).

Paley graphs may be generalized in several ways. One such variation of Paley graphs was given in [2]. The vertices are the elements of $\mathrm{GF}(q)$, with q a prime power. A *cubic Paley graph* $P_q^{(3)}$ of order $q \equiv 1 \pmod 3$

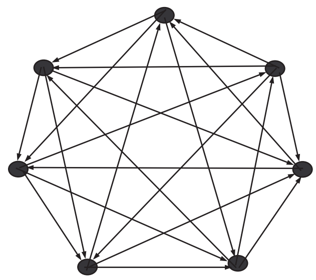

FIGURE 3. The unique 2-e.c. tournament of minimum order, $\overrightarrow{P_7}$.

has vertices joined if their difference is the cube of an element of GF(q). A *quadruple Paley graph* $P_q^{(4)}$ of order $q \equiv 1 \pmod 8$ has vertices joined if their difference is the fourth power of an element of GF(q). The following result from [2] was proved using character sum estimates.

Theorem 6. (1) *If* $q > (2n2^{2n-1} - 2^{2n} + 1)2^n\sqrt{q} + 3n2^{-n}3^{2n-1}$, *then* $P_q^{(3)}$ *is n-e.c.*
(2) *If* $q > (2n2^{2n-1} - 2^{2n} + 1)3^n\sqrt{q} + 4n3^{-n}4^{2n-1}$, *then* $P_q^{(4)}$ *is n-e.c.*

Another recent variation on Paley graphs was given in [30]. Let $q = p^r$ be a prime power so that $q \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$. Let v be a generator of the multiplicative group of GF(q) (hence, v is a primitive root of q). Define the graph $P^*(q)$ to have vertices the elements of GF(q), with two vertices joined if their difference is of the form v^j where $j \equiv 0$ or $j \equiv 1 \pmod 4$. Similar to Paley graphs, any graph $P^*(q)$ is strongly regular, self-complementary, and symmetric. Using a character sum estimate, the following result is proven in [30].

Theorem 7. *If* $q = p^r$ *is a prime power so that* $q \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$, *and* $q > 8n^2 2^{8n}$, *then* $P^*(q)$ *is n-e.c.*

4. Matrices and Combinatorial designs

As described in the last section, the proofs of adjacency properties of Paley graphs and their generalizations exploit non-elementary character sum estimates similar to Theorem 5. A different approach using combinatorial matrix theory was given in [11]. A *Hadamard matrix* is a square matrix whose entries are either $+1$ or -1 and whose rows are mutually orthogonal. As is well-known, the order of a Hadamard matrix must be 1, 2, or a multiple of 4.

Let J_n be the $n \times n$ matrix of all 1s. A Hadamard matrix H of order $4n^2$ is a *Bush-type Hadamard matrix* if $H = [H_{ij}]$, where H_{ij} are submatrices (or blocks) of order $2n$, $H_{ii} = J_{2n}$, and $H_{ij}J_{2n} = J_{2n}H_{ij}$, for $i \neq j$, $1 \leq i \leq 2n$, $1 \leq j \leq 2n$. A symmetric Bush-type Hadamard matrix of order 4 is shown below.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

A symmetric Bush-type Hadamard matrix of order $4n^2$ is the $\mp$ adjacency matrix (with -1 for adjacency, $+1$ for non-adjacency) of a $\mathrm{SRG}(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$. By using a result of Kharaghani which generates a Bush-type Hadamard matrix of order $16n^2$ from any Hadamard matrix of order $4n$, the following result is proved in [11].

Theorem 8. *Let* $4n$ *be the order of a Hadamard matrix, with* $n > 1$ *odd. Then there is a* 3*-e.c.* $\mathrm{SRG}(16n^2, 8n^2 - 2n, 4n^2 - 2n, 4n^2 - 2n)$.

While this construction was promising, it unfortunately did not generate n -e.c. graphs with $n > 3$. Cameron and Stark [17] provided a considerable breakthrough by giving a new explicit family of strongly regular n-e.c. graphs that are not isomorphic to Paley graphs.

To define the graphs of [17] we require some facts from design theory. Let v and λ be fixed positive integers, and fix k so that $2 \leq k < v$. Let $S = \{1, 2, \ldots, v\}$. A 2-*(v, k, λ) design* (or simply 2-*design*) is a collection D of subsets of S called *blocks* such that

(1) each block has exactly k elements;
(2) each 2-element subset of S is contained in exactly λ blocks.

An *affine design* is a 2-design such that

(1) every two blocks are disjoint or intersect in a constant number of points;
(2) each block together with all blocks disjoint from it form a *parallel class*: set of n mutually disjoint blocks partitioning all points of the design.

Following results by Fon-der-Flaass on generating strongly regular graphs from affine designs, the following construction was given in [17]. Fix q a prime power such that $q \equiv 3 \pmod 4$. For the Paley tournament $\overrightarrow{P_q}$, let A_q be the adjacency matrix of $\overrightarrow{P_q}$ with $(A_q)_{i,j} = 1$ if (i, j) is a directed edge, and $(A_q)_{i,j} = -1$ if (j, i) is a directed edge (and 0 otherwise). Let I_q be the order q identity matrix, $B_q = A_q - I_q$, and let C_q be the order $(q + 1)$ square matrix obtained from B_q by adding an initial row of 1's and a column of 1 's. For each q, the last q rows of C_q are the ± 1 incidence matrix of an affine design. Each parallel class contains two blocks, corresponding to $+$ and $-$. Let D_q be the incidence matrix of a design on $q+1$ points with vertices corresponding to columns of C_q, and parallel classes corresponding to rows of C_q.

Choose permutations π_i, $1 \leq i \leq q+1$ independently and uniformly at random from the set of all permutations on $\{1, 2, \ldots, q\}$. Let $\mathcal{S}_1, \ldots, \mathcal{S}_{q+1}$ be affine designs such that the point sets $V_1, \ldots, V_{q+1}$ are copies of $\{1, 2, \ldots, q+1\}$ and such that the jth row of M_i is the $\pi_i(j)th$ row of D_q. Let $\mathcal{S}_i = (V_i, L_i)$, and set $\mathcal{I} = \{1, 2, \ldots, q+1\}$.

For every i, denote the parallel class of $\mathcal{S}_i$ corresponding to the jth row of M_i by symbols $\mathcal{L}_{ij}$, $j \in \mathcal{I}\backslash\{i\}$. For $v \in V_i$, let $l_{ij}(v)$ denote the line in the parallel class $\mathcal{L}_{ij}$ which contains v. For every pair i, j with $i \neq j$ choose an arbitrary bijection $\sigma_{i,j} : \mathcal{L}_{ij} \to \mathcal{L}_{ji}$ from the two possibilities. It is required that $\sigma_{j,i} = \sigma_{i,j}^{-1}$.

Define a graph $G_1((\mathcal{S}_i), (\sigma_{i,j}))$ with vertices $X = \bigcup_{i \in I} V_i$. The sets V_i are independent. Two vertices $u \in V_i$ and $v \in V_j$ with $i \neq j$ are joined if $w \in \sigma_{i,j}(l_{ij}(v))$. For sufficiently large q, it is proved in [17] that $G_1((\mathcal{S}_i), (\sigma_{i,j}))$ is n-e.c. Further, the construction supplies a *prolific* or superexponential number of non-isomorphic examples.

Theorem 9. *Suppose that q is a prime power such that $q \equiv 3 \ (mod\ 4)$. There is a function $\varepsilon(q) = O(q^{-1} \log q)$ such that there exist $2^{\binom{q+1}{2}(1-\varepsilon(q))}$ non-isomorphic* $\mathrm{SRG}((q+1)^2, q(q+1)/2, (q^2-1)/4, (q^2-1)/4)$ *which are n-e.c. whenever $q \geq 16n^2 2^{2n}$.*

A drawback of Theorem 9 is that its proof uses Poisson approximation theory and is not elementary. In the next section, a geometric construction of n-e.c. graphs bypasses this difficulty (but does not provide a prolific number of examples).

4.1. **Steiner triple systems.** Another direction of research concerns n-e.c. graphs arising from Steiner triple systems. A *Steiner triple system of order v*, written $\mathrm{STS}(v)$ is a 2-$(v, 3, 1)$ design. An $\mathrm{STS}(v)$ exists if and only if $v \equiv 1$ or $v \equiv 3 \pmod 6$. The *block-intersection graph of a Steiner triple system* is a graph whose vertices are the blocks of the $\mathrm{STS}(v)$ with two blocks joined if they have non-empty intersection. Such a graph is an $\mathrm{SRG}(\frac{v^2-v}{6}, \frac{3v-9}{2}, \frac{v+3}{2}, 9)$. In [24] it is shown that Steiner triple systems with a 3-e.c. block intersection graph are rare.

Theorem 10. (1) *The block-intersection graph of a* $\mathrm{STS}(v)$ *is 2-e.c. if and only if $v \geq 13$.*

(2) *If* STS(v) *has a* 3*-e.c. block-intersection graph, then* $v = 19$ *or* $v = 21$.

A computer search in [24] found 3-e.c. graphs when $v = 19$, but none if $v = 21$ (although they may exist). McKay and Pike [33] found 2-e.c. graphs using block-intersection graphs of balanced incomplete block designs. It is an open problem to construct n-e.c. graphs for $n > 3$ using block-intersection graphs of designs.

4.2. **Matrices and constraints.** A recent construction of n-e.c. graphs using matrices was given in [5], based on certain constructions in combinatorial set theory due to Hausdorff [27]. Consider matrices with $r = 2n(n-1)+1$ rows and c columns, where c is chosen so that

$$2^c \geq 2^{r^2}\binom{rc}{n-1}.$$

We define a graph G as follows. The vertices of G are $r \times c$ zero-one matrices, with the property that at least $n(n-1)+1$ rows are identical and equal to the vector v. The edges are described as follows. A *constraint* is given by a pair (A, F), where A is a set of $n-1$ entries in an $r \times c$ matrix, and F is a family of at most n functions from A to $\{0, 1\}$. A vertex V *satisfies* a constraint (A, F) if for some $f \in F$, for all $(i, j) \in A$, $V_{ij} = f(i, j)$. Fix a surjection C from the set of c-component zero-one vectors onto the set of constraints (such a C exists by a counting argument). Every vertex V determines a constraint $V^* = C(v)$. There is a directed edge (V, W) if W satisfies the constraint V^*. Let V be joined to W if (V, W), (W, V) are either both present or neither is present.

The graph G has parameters r, c, v, and the surjection C. Hence, we will we use the notation $G(r, c, v, C)$ for such a graph. The following is proved in [5] without using the probabilistic method.

Theorem 11. *For a fixed* n, *let* $r = 2n(n-1)+1$, c *satisfy* $2^c \geq 2^{r^2}\binom{rc}{n-1}$, *and let* v *and* C *be chosen as above. Then the graph* $G(r, c, v, C)$ *is* n*-e.c.*

5. Finite geometry

The last constructions of n-e.c. graphs we supply use finite geometry. The geometric approach, apart from being somewhat more intuitive than the previous ones exploiting designs and matrices, will supply elementary constructions and proofs.

Consider an *affine plane* A *of order* q, where A is coordinatized over GF(q), with q a prime power, In particular, A is a 2-$(q^2, q, 1)$ design (with blocks called *lines*), and hence, satisfies the property that given a point x and a line ℓ, there is a unique line parallel to ℓ that goes through x. Each line contains exactly q points. The relation of parallelism on the set of lines is an equivalence relation, and the equivalence classes are called *parallel classes*. There are $q+1$ parallel classes (corresponding to points on the lines at infinity). We use the notation pq for the line between p and q. (Although this notation conflicts with our earlier notation for edges of a graph, we keep both notations since they are standard.)

We now consider a construction of strongly regular graphs which is due to Delsarte and Goethals, and to Turyn; see [35]. Let ℓ_∞ be the $(q+1)$-element line at infinity, identified with slopes. Fix $S \subseteq \ell_\infty$. Define $G(q,S,A)$ to have vertices the points of A, and two vertices p and q are joined if and only if the line pq has slope in S. It is easy to see that $G(q,S,A)$ is a $\mathrm{SRG}(q^2, |S|(q-1), q-2+(|S|-1)(|S|-2), |S|(|S|-1))$. Let $\mathcal{G}(q,A)$ be the family of graphs $G(q,S,A)$ for all choices of S; if $0 \leq k \leq q+1$ is fixed, then write $\mathcal{G}(q,k,A)$ for the subfamily of all graphs in $\mathcal{G}(q,A)$ where $|S| = k$. For a fixed k, $\mathcal{G}(q,k,A)$ may contain non-isomorphic members (in general, this is nontrivial to determine; see [16]).

Fix A, an affine plane of even order $q \geq 8$ coordinatized by $\mathrm{GF}(2^k)$ (hence, A is Desarguesian). Choose S to be any fixed set of $\frac{q}{2}$ slopes in l_∞. It follows that G is a $\mathrm{SRG}(q^2, \frac{q(q-1)}{2}, \frac{q(q-2)}{4}, \frac{q(q-2)}{4})$ (which is a *Latin square graph*). For infinitely many values of q, it is an exercise to show that the parameters for the graphs in $\mathcal{G}(q, \frac{q}{2}, A)$ are different than those in Theorem 9. The graphs in $\mathcal{G}(q, \frac{q}{2}, A)$ are well-known examples of quasi-random graphs; see [20].

We may consider $\mathcal{G}(q, \frac{q}{2}, A)$ as an equiprobable probability space of cardinality $\binom{q+1}{\frac{q}{2}}$: each point of the probability space corresponds to a choice of S with $|S| = \frac{q}{2}$. With this view, the following result was proven in [3].

Theorem 12. *Let q be a power of 2 and fix n a positive integer. With probability 1 as $q \to \infty$, $\mathcal{G}(q, \frac{q}{2}, A)$ is n-e.c.*

The proof of Theorem 12 is the first elementary one that an explicit family has strongly regular n-e.c. members for all positive integers n. The approach is, however, randomized. Not all the graphs $\mathcal{G}(q, \frac{q}{2}, A)$ are n-e.c. even for large q as described in the following result from [3].

Theorem 13. *Let $q \geq 8$ be a power of 2.*

1. *All graphs $G \in \mathcal{G}(q, \frac{q}{2}, A)$ are 3-e.c.*
2. *For all $n \geq 4$, there is a $G \in \mathcal{G}(q, \frac{q}{2}, A)$ that is not n-e.c.*

We now give a new construction of explicit n-e.c. graphs. Our method generates strongly regular n-e.c. graphs. Let A be an affine plane with q^2 points. For a fixed $p \in (0,1)$, choose $m \in \ell_\infty$ to be in S independently with probability p; with the remaining probability, m is in the complement of S. This makes $\mathcal{G}(q,A)$ into a probability space which we denote $\mathcal{G}_p(q,A)$. While $|S|$ is a random variable, all choices of S give rise to a strongly regular graph. We prove the following result.

Theorem 14. *Fix $p \in (0,1)$ and n a positive integer. With probability 1 as $q \to \infty$, $\mathcal{G}_p(q,A)$ is n-e.c.*

Proof. Fix disjoint sets of vertices X and Y in G, with $|X \cup Y| = n$. Let $U = X \cup Y$. We prove that for large q, with probability 1 there is a vertex z correctly joined to X and Y. To accomplish this, we construct a set P_U of points, disjoint from U, such that with probability 1, z is in P_U. We set $s = \lceil q^b \rceil$, where $b < 1$ is fixed.

Fix a point v of A. The *projection from v onto l_∞*, is the map $\pi_v : A\backslash\{v\} \to \ell_\infty$ taking a point x to the intersection of vx with ℓ_∞. Hence, $\pi_v(x)$ is the slope of the line vx. If V is a set of points, then let $\pi_v(V) = \bigcup_{x\in V} \pi_v(x)$.

For sufficiently large q, we inductively construct a set of points P_U distinct from U with the following properties.

(1) If $p \in P_U$, then $|\pi_p(U)| = n$.
(2) For all distinct p and q in P_U, $\pi_p(U) \cap \pi_q(U) = \emptyset$.
(3) $|P_U| = s$.

Define $P_{U,1}$ by choosing any point $p_1 \notin U$ that is not on a line joining two points of U. For large q

$$n + \binom{n}{2}(q-2) < q^2,$$

so we may find such a p_1.

For a fixed positive $i \leq s-1$, suppose that $P_{U,i}$ has been constructed for large q, with $P_{U,i}$ containing $P_{U,1}$, and $|P_{U,i}| = i$. We would like to choose $p_{i+1} \notin U$ to be a point that is

(i) not on a line joining two points of U, and
(ii) not on a line joining a point of U to a point in $\bigcup_{j=1}^{i} \pi_{p_j}(U)$.

Condition i) rules out $\binom{n}{2}$ lines, while ii) rules out $ni + n(n-1)i$ lines. For large q

$$n + \binom{n}{2}(q-2) + ni(q-1) + n(n-1)i(q-2) < n^2 q^{b+1} < q^2,$$

so we may find a suitable p_{i+1} satisfying items 1) and 2). Add p_{i+1} to $P_{U,i}$ to form $P_{U,i+1}$. Define $P_U = \bigcup_{i=1}^{s} P_{U,i}$ so $|P_U| = s$.

For a fixed $U = X \cup Y$, we estimate the probability that none of the vertices of P_U are correctly joined to X and Y. Suppose that $m = |X|$ and $n - m = |Y|$. By item 1), note that any z in P_U has the property that zx and zy have distinct slopes, where x, y are distinct points of U. Note also that zx is an edge of G if and only if $\pi_z(x) \in S$. Therefore, the probability that a given z in P_U is not joined correctly to X and Y is the positive constant

$$p_n = 1 - p^m(1-p)^{n-m}. \tag{5.1}$$

By item (2) in the defining properties of P_U, any two distinct points of P_U induce disjoint slope sets in ℓ_∞. In particular, the probability (5.1) independently holds for any choice of z in P_u. Hence, the probability that no z in P_U is correctly joined to X and Y is $(p_n)^{\lceil q^b \rceil}$. The probability that $\mathcal{G}_p(q, A)$ is not n-e.c. is therefore at most

$$\binom{q^2}{n} 2^n (p_n)^{\lceil q^b \rceil} = O(n \log(2q^2) + q^b \log(p_n)),$$

which tends to 0 as $q \to \infty$. □

We note first that this randomized construction of n-e.c. graphs may be generalized to higher dimensional affine spaces. The line at infinity is replaced by the hyperplane H_∞ at infinity (whose points are vectors of co-dimension 1). We may choose vectors in H_∞ independently with fixed probability to obtain a probability space of regular (though not necessarily strongly regular) graphs over the affine space. An n-e.c. result analogous to Theorem 14 holds for these graphs; the details will appear in a forthcoming paper.

As a second remark concerning Theorem 14, if elements of S are chosen independently with probability p, then the expected cardinality of S is $p(q+1)$. By the Chernoff bounds (see, for example, Section 2.1 of [29]), with probability tending to 1 as n tends to infinity, a graph in $\mathcal{G}_p(q, A)$ is strongly regular with degree concentrated around $p(q+1)$. Hence, if $p = 1/2$ with q a power of 2, the space $\mathcal{G}_p(q, A)$ has distribution similar to the space $\mathcal{G}(q, \frac{q}{2}, A)$. If $p \neq 1/2$, then with high probability, graphs in $\mathcal{G}_p(q, A)$ are n-e.c. and have different expected strongly regular parameters than the graphs in $\mathcal{G}(q, \frac{q}{2}, A)$, or Paley graphs.

We finish with a variant on the problem of finding explicit n-e.c. graphs. For triangle-free graphs, the natural version of the n-e.c property requires that the set A must be independent. To be more precise, a graph is *n-saturated* if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B can be empty) such that A is independent, there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. The countable universal homogeneous triangle-free graph is n-saturated for all n, but it is an open problem whether finite n-saturated triangle-free graphs exist for all n. See [18] for further background on this problem. For a related property of triangle-free graphs, see [1].

The existence of finite n-saturated graphs is an important problem in the first-order logic of discrete structures: it is related to the problem involving the approximation of infinite structures by finite ones. Probabilistic arguments (which proved useful in the existence proofs of n -e.c. graphs) do not seem to apply. It can be shown that the Higman-Sims graph (the unique isomorphism type of $\mathrm{SRG}(100, 22, 0, 6)$) is 3 -saturated, but no strongly regular 4-saturated triangle-free graphs exist.

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