Growth deletion models for the web graph and other massive networks

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Abstract: We propose new evolutionary stochastic models for the web graph and other massive networks, where edges are deleted over time and an edge is chosen to be deleted with probability inversely proportional to the in-degree of the destination. The degree distributions of graphs generated by our models follow a power law. A rigorous proof of power law degree distributions is given using martingales and concentration results. Depending on the parameters, the exponent of the power law can be any number in $(1, \infty)$. For this reason, our models apply not only to the web graph, but to certain biological networks, where the power law exponent is in the interval $(1, 2)$.

Keywords. web graph, power law graphs, degree distributions, scale free networks, stochastic graph models

1 Introduction

In the past few years, there has been much interest in understanding the properties of real-world large-scale networks such as the structure of the Internet and the World Wide Web. It has been observed that many such networks have a so-called *power law degree distribution*: the proportion of nodes of degree k is approximately $\frac{1}{k^\gamma}$, where $\gamma > 1$ is a fixed real number. Such graphs are sometimes called *scale-free* in the literature. A graph is called a *power law graph* if the fraction of nodes with degree k is proportional to $\frac{1}{k^\gamma}$ for some constant $\gamma > 0$. The standard models of random graphs introduced by Erdős and Rényi [11] and Gilbert [12] are not appropriate for studying these networks, since they generate graphs which, with high probability, have binomial degree distributions.

A large number of power law random graph models [1, 3, 5, 10, 13] have been proposed. For two recent surveys on models of the web graph, see [4, 6]. In all of these models, at

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each time step nodes and edges are added, but never *deleted*. An evolving graph model incorporating in its design both the addition and deletion of nodes and edges may more accurately model the evolution of the web graph. Recently, the models of [7, 9] incorporate the addition and deletion of nodes *during* the generation of nodes. We refer to such models as *growth-deletion models*.

In [7], Chung, Lu introduced a growth-deletion model $G(p_1, p_2, p_3, p_4, m)$, for undirected graphs with parameters m a positive integer, and probabilities p_1, p_2, p_3, p_4 satisfying $p_1 + p_2 + p_3 + p_4 = 1$, $p_3 < p_1$, and $p_4 < p_2$. Let H be a fixed nonempty graph, and let $G_0 = H$. To form $G_{t+1,}$ they proceed as follows. With probability p_1, add a new node v_{t+1} and m edges from v_{t+1} to existing nodes chosen with probability proportional to their degrees. We refer to this as *preferential attachment*. With probability p_2, add m new edges with endpoints to be chosen among existing nodes by preferential attachment. With probability p_3, delete a node chosen uniformly at random $(u.a.r)$. With probability p_4, delete m edges chosen u.a.r.

In [9], Cooper et al. introduced an undirected model with three parameters α, α_0 and α_1 which generates a sequence of simple graphs $G_t, t = 1, 2, ...,$ where the graph $G_t = (V_t, E_t)$ has v_t nodes and e_t edges. Start with G_1 consisting of an isolated node x_1. At time t, with probability $1 - \alpha - \alpha_0$, delete a randomly chosen node x from V_{t-1}. If $V_{t-1} = \emptyset$, then do nothing; with probability α_0, delete $\min\{m, |E_{t-1}|\}$ randomly chosen edges from E_{t-1}; with probability α_1, add a node x_t with m random edges incident with x_t to G_{t-1}, the endpoints are chosen by preferential attachment; with probability $\alpha - \alpha_1$, add m random edges to existing nodes. The endpoints are chosen by preferential attachment. The models of [7, 9] generate scale-free graphs whose exponent γ is in the interval $[2, \infty)$.

However, we observe that in real-world networks such as the web graph, new nodes are more likely to join existing nodes with high degree, while links pointing to a node with high degree are less likely to be deleted. Motivated by this observation, we propose a new directed model, called a *biased edge-deletion model*, where we delete a directed edge with probability inversely proportional to the in-degree of the destination. Hence, edges pointing to "popular" nodes (that is, nodes with high in-degree) are less likely to be deleted.

We describe these network models precisely in Section 2, and state our main results there. In Section 3, a rigorous proof for the power law degree distributions is given using martingales and concentration results. We emphasize that our models generate graphs with power law exponent in the interval $(1, \infty)$. This is significant since certain massive networks, such as the network of protein-protein interaction networks in a living cell, have power law exponents in the interval $(1, 2)$; see [8]. Hence, our models are used not only as models of the web graph, but for many other massive networks.

2 Edge-deletion models

In this section, we introduce three edge-deletion models and state our main result. Assuming that α and β are two nonnegative real numbers satisfying $\alpha + \beta < 1$ and $\beta < \frac{1}{2}$, we consider a random process which generates a sequence of graphs $G_t, t = 0, 1, 2, ...,$. The graph $G_t = (V_t, E_t)$ will have n_t nodes and e_t edges.

Model 1: To initialize the process, at $t = 0$ we start with an initial digraph G_0 with n_0 nodes and m_0 edges.

At time t, with probability $1 - \alpha - \beta$ we add a node v_t to G_{t-1}, with a directed loop. With probability α we add a directed edge uv to the existing nodes, where the origin is chosen with probability proportional to its out-degree and the destination is chosen proportional to its in-degree. With probability β, if $e_{t-1} > 0$, we delete a directed edge, where an edge is chosen inversely proportional to the in-degree of the destination; if $e_{t-1} = 0$, then we do nothing.

Model 2: This model is defined similarly to Model 1 except that edges to be deleted u.a.r.

The next model generates undirected graphs.

Model 3: To initialize the process, at $t = 0$ we start with an initial graph G_0 with n_0 nodes and m_0 edges.

At time t, with probability $1-\alpha-\beta$ we add a node v_t to G_{t-1} along with an edge. An endpoint is v_t, the other endpoint is chosen by preferential attachment. With probability α we add an edge uv to the existing nodes. The endpoints u and v are chosen by preferential attachment. With probability β, if $e_{t-1} > 0$, we delete an edge u.a.r; if $e_{t-1} = 0$, then we do nothing.

Note: Model 3 is the same as those models in [7] with $p_3 = 0$ and $m = 1$, and in [9] with $\alpha + \alpha_0 = 1$ and $m = 1$. We obtain the same result as those in [7, 9].

Denote by $d^{in}_{k,t}$ the number of nodes with in-degree k at time t in Model 1 and Model 2, and denote by $d_{k,t}$ the number of nodes with degree k at time t in Model 3. In the following theorem, we will show that, asymptotically, $d^{in}_{k,t}$ and $d_{k,t}$ follow a power law. If A is an event in a probability space, then we write $Pr(A)$ for the probability of A in the space.

Theorem 1 *For the models 1, 2, and 3, we have the following.*

1. *For Model 1, the in-degree distribution follows a power law with exponent* $\gamma = 1 + \frac{1-2\beta}{\alpha} \in (1, \infty)$. *More precisely, we have*

$$Pr\left(\left|d^{in}_{k,t} - \frac{b_{k,1}}{1-\alpha-\beta}n_t\right| > 2\epsilon\sqrt{t}\left(1 + \frac{b_{k,1}}{1-\alpha-\beta}\right)\right) < 4e^{-\epsilon^2/2},$$

where

$$b_{k,1} = (1 + O(k^{-1}))C_1(\alpha, \beta)k^{-\gamma},$$

and $C_1(\alpha, \beta)$ *is a constant.*

2. *For Model 2, the in-degree distribution follows a power law with exponent* $\gamma = 1 + \frac{1-2\beta}{\alpha-\beta} \in (1, \infty)$. *More precisely,*

   $$Pr\left(\left|d_{k,t}^{in} - \frac{b_{k,2}}{1-\alpha-\beta}n_t\right| > 2\epsilon\sqrt{t}\left(1 + \frac{b_{k,2}}{1-\alpha-\beta}\right)\right) < 4e^{-\epsilon^2/2},$$

   where

   $$b_{k,2} = (1 + O(k^{-1}))C_2(\alpha, \beta)k^{-\gamma},$$

   and $C_2(\alpha, \beta)$ *is a constant.*

3. *For Model 3, the degree distribution follows a power law with exponent* $\gamma = 1 + \frac{2-4\beta}{1+\alpha-3\beta} \in (1, \infty)$. *More precisely,*

   $$Pr\left(\left|d_{k,t} - \frac{b_{k,3}}{1-\alpha-\beta}n_t\right| > 2\epsilon\sqrt{t}\left(1 + \frac{b_{k,3}}{1-\alpha-\beta}\right)\right) < 2(e^{-\epsilon^2/2} + e^{-\epsilon^2/8}),$$

   where

   $$b_{k,3} = (1 + O(k^{-1}))C_3(\alpha, \beta)k^{-\gamma},$$

   and $C_3(\alpha, \beta)$ *is a constant.*

3 Proof of Theorem 1

As t increases, G_t may be defined recursively. For each t, let τ_t be a random variable of G_t. Let c be a positive integer. The random variable τ_t is said to satisfy the c-*Lipschitz condition* if

$$|\tau_{t+1}(G_{t+1}) - \tau_t(G_t)| \leq c$$

whenever G_{t+1} is obtained from G_t by adding or deleting some edges or some nodes at time $t + 1$. The proof of Theorem 1 will follow by the next Lemma, which is the Azuma-Hoeffding Inequality. See for example, Theorem 7.2.1 of [2]. If X is a random variable, then we denote $E(X)$ for its expected value.

Lemma 2 *If* τ_t *satisfies the c-Lipschitz condition, then for every* $\delta > 0$

$$Pr[|\tau_t - E(\tau_t)| > \delta\sqrt{t}] < 2e^{-\frac{\delta^2}{2c^2}}.$$

In particular, τ_t is almost surely very close to its expected value $E(\tau_t)$ with an error term $o(t^{\frac{1}{2}+\delta})$ for any $\delta > 0$, as t approaches infinity.

The next two lemmas are useful in our proof of Theorem 1, and also serve as a warm up for the application of Lemma 2.

Lemma 3 *For each of the models 1, 2, and 3, we have the following.*

1. *For $t \geq 0$, the expected value of the number of (directed)edges e_t at time t is*

$$E(e_t) = m_0 + (1 - 2\beta)t + o(t),$$

where $o(t)$ is a lower order term.

2. *For every $\epsilon > 0$,*

$$Pr[|e_t - E(e_t)| > \epsilon t^{\frac{2}{3}}] < 2e^{-\frac{\epsilon^2}{2}t^{\frac{1}{3}}}.$$

Proof Define

$$Y_t = \begin{cases} 1 & \text{an edge is added at time } t; \\ -1 & \text{an edge is deleted at time } t; \\ 0 & \text{otherwise.} \end{cases}$$

So, $e_{t+1} = e_t + Y_t$. Hence, $E(e_{t+1}) = E(e_t) + E(Y_t)$. By simple calculations, we obtain that

$$E(E(Y_t|e_t \neq 0)) = 1 - 2\beta,$$

and

$$E(E(Y_t|e_t = 0)) = 1 - \beta.$$

So,

$$\begin{aligned} E(Y_t) &= E(E(Y_t|e_t \neq 0))Pr(e_t \neq 0) + E(E(Y_t|e_t = 0))Pr(e_t = 0) \\ &= (1 - 2\beta)Pr(e_t \neq 0) + (1 - \beta)Pr(e_t = 0) \\ &= 1 - 2\beta + \beta Pr(e_t = 0). \end{aligned}$$

So,

$$E(e_{t+1}) = E(e_t) + 1 - 2\beta + \beta Pr(e_t = 0). \tag{3.1}$$

Let S_t be the number of time steps in which an edge is added from time 1 to time t. At each time step, with probability $1-\alpha-\beta+\alpha = 1-\beta$ we add an edge, so $E(S_t) = (1-\beta)t$. Since S_t satisfies the 1-Lipschitz condition, by Lemma 2, for every $\delta > 0$

$$Pr[|S_t - (1 - \beta)t| > \delta\sqrt{t}] < 2e^{-\frac{\delta^2}{2}}. \tag{3.2}$$

Obviously, the number of time steps in which an edge is deleted (if there are any edges) equals $t - S_t$. Therefore, $e_t \geq m_0 + S_t - (t - S_t)$, and thus $e_t = 0$ implies that $S_t \leq \frac{t-m_0}{2}$. So,

$$\begin{aligned} Pr\,(e_t = 0) &\leq Pr\left(S_t \leq \frac{t - m_0}{2}\right) \\ &\leq Pr\left(|S_t - (1 - \beta)t| > (\frac{1}{2} - \beta)t + \frac{m_0}{2}\right). \end{aligned} \tag{3.3}$$

By (3.3) and (3.2) with $\delta = (\frac{1}{2} - \beta)\sqrt{t} + \frac{m_0}{2\sqrt{t}}$, we obtain that

$$Pr(e_t = 0) \leq 2e^{-\frac{1}{2}(\frac{1}{2}-\beta)^2 t - \frac{m_0^2}{8t} - \frac{m_0(\frac{1}{2}-\beta)}{2}}. \tag{3.4}$$

Note that $E(e_0) = m_0$. By (3.1) and (3.4), we obtain that

$$E(e_t) = m_0 + (1 - 2\beta)t + o(t),$$

where $o(t)$ is a lower order term. Since e_t satisfies the 1-Lipschitz condition, by Lemma 2 with $\delta = \epsilon t^{1/6}$, (2) holds. $\blacksquare$

Lemma 4 *For each of the models 1, 2, and 3, we have the following.*

1. *For $t \geq 0$, the expected number of nodes n_t at time t is*

$$E(n_t) = n_0 + (1 - \alpha - \beta)t.$$

2. *For every $\epsilon > 0$,*

$$Pr(|n_t - E(n_t)| > \epsilon\sqrt{t}) < 2e^{-\frac{\epsilon^2}{2}}.$$

The proof is similar to the proof of Lemma 3, so it is omitted.

Proof of Theorem 1. We prove only (1) due to space limitations. The full proof of (2) and (3) will appear in a forthcoming journal paper. For the sequence of random variables $\{d_{k,t}^{in}\}$, we will compute the corresponding expected value $E(d_{k,t}^{in})$ here. At time 0, there is an initial graph G_0 with n_0 nodes and m_0 edges. Let $d_{k,0}^{in} = d_k^0$ be the number of nodes with in-degree $k, k \geq 0$ at time 0. At time 1, a node with a loop is added. We abbreviate "with probability" by "w.p.". Assume that there are e_t edges at time t, for $t \geq 0$. It is not hard to see that

$$d_{0,t+1}^{in} = \begin{cases} d_{0,t}^{in} + 1 & \text{w.p. } \beta\frac{d_{1,t}^{in}}{e_t}; \\ d_{0,t}^{in} & \text{otherwise.} \end{cases}$$

and

$$d_{1,t+1}^{in} = \begin{cases} d_{1,t}^{in} + 1 & \text{w.p. } 1 - \alpha - \beta + \beta\frac{d_{2,t}^{in}}{e_t}; \\ d_{1,t}^{in} - 1 & \text{w.p. } (\alpha + \beta)\frac{d_{1,t}^{in}}{e_t}; \\ d_{1,t}^{in} & \text{otherwise.} \end{cases}$$

In general, for $k > 1$, $d_{k,t+1}^{in}$ can increase by 1 because a node of in-degree $k-1$ receives an edge or a node of in-degree $k+1$ loses an edge; $d_{k,t+1}^{in}$ can decrease by 1 because a node of in-degree k receives an edge or loses an edge. Thus we have that

$$d_{k,t+1}^{in} = \begin{cases} d_{k,t}^{in} + 1 & \text{w.p. } \alpha\frac{(k-1)d_{k-1,t}^{in}}{e_t} + \beta\frac{d_{k+1,t}^{in}}{e_t}; \\ d_{k,t}^{in} - 1 & \text{w.p. } \alpha\frac{kd_{k,t}^{in}}{e_t} + \beta\frac{d_{k,t}^{in}}{e_t}; \\ d_{k,t}^{in} & \text{otherwise.} \end{cases}$$

Hence,

$$E(d^{in}_{k,t+1}|G_t) = d^{in}_{k,t}\left(1 - \frac{k\alpha+\beta}{e_t}\right) + \alpha\frac{(k-1)d^{in}_{k-1,t}}{e_t} + \beta\frac{d^{in}_{k+1,t}}{e_t}. \tag{3.5}$$

Define $\overline{e}_t = m_0 + (1-2\beta)t + o(t)$ where $o(t)$ is a lower order term and let $A_t = \{|e_t - \overline{e}_t| \leq \epsilon t^{\frac{2}{3}}\}$ be an event. Define

$$a_t = \begin{cases} 1 & A_t \text{ occurs;} \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3, we know that

$$Pr(a_t = 1) \geq 1 - 2e^{-\frac{\epsilon^2 t^{\frac{1}{3}}}{2}}. \tag{3.6}$$

By (3.5), we obtain that

$$d^{in}_{k,t}\left(1 - \frac{k\alpha+\beta}{\overline{e}_t - \epsilon t^{\frac{2}{3}}}\right) + \frac{\alpha(k-1)d^{in}_{k-1,t} + \beta d^{in}_{k+1,t}}{\overline{e}_t + \epsilon t^{\frac{2}{3}}} \leq E(d^{in}_{k,t+1}|G_t, a_t = 1)$$

$$\leq d^{in}_{k,t}\left(1 - \frac{k\alpha+\beta}{\overline{e}_t + \epsilon t^{\frac{2}{3}}}\right) + \frac{\alpha(k-1)d^{in}_{k-1,t} + \beta d^{in}_{k+1,t}}{\overline{e}_t - \epsilon t^{\frac{2}{3}}}. \tag{3.7}$$

It is easy to see that

$$\begin{aligned} E(d^{in}_{k,t+1}|G_t) &= P(a_t = 1)E(d^{in}_{k,t+1}|G_t, a_t = 1) + \\ &\quad P(a_t = 0)E(d^{in}_{k,t+1}|G_t, a_t = 0). \end{aligned} \tag{3.8}$$

Note that $d^{in}_{k,t} - 1 \leq E(d^{in}_{k,t+1}|G_t) \leq d^{in}_{k,t} + 1$. So,

$$E(d^{in}_{k,t}) - 1 \leq E(d^{in}_{k,t+1}) \leq E(d^{in}_{k,t}) + 1. \tag{3.9}$$

Taking expectation on both sides of (3.8), together with (3.6), (3.7) and (3.9), we obtain that

$$\left(E(d^{in}_{k,t})\left(1 - \frac{k\alpha+\beta}{\overline{e}_t - \epsilon t^{\frac{2}{3}}}\right) + \frac{\alpha(k-1)E(d^{in}_{k-1,t}) + \beta E(d^{in}_{k+1,t})}{\overline{e}_t + \epsilon t^{\frac{2}{3}}}\right)\left(1 - 2e^{-\frac{\epsilon^2 t^{\frac{1}{3}}}{2}}\right)$$

$$\leq E(d^{in}_{k,t+1}) \leq (E(d^{in}_{k,t}) + 1)\left(2e^{-\frac{\epsilon^2 t^{\frac{1}{3}}}{2}}\right) +$$

$$E(d^{in}_{k,t})\left(1 - \frac{k\alpha+\beta}{\overline{e}_t + \epsilon t^{\frac{2}{3}}}\right) + \frac{\alpha(k-1)E(d^{in}_{k-1,t}) + \beta E(d^{in}_{k+1,t})}{\overline{e}_t - \epsilon t^{\frac{2}{3}}}. \tag{3.10}$$

Let $E(d_{k,t}^{in}) = b_{k,1}t + c_{k,t}$, where $c_{k,t} = o(t)$ is a lower order term. To choose an appropriate value for $b_{k,1}$, we substitute it into (3.10) and let t tend to infinity. We obtain, for $k > 1$,

$$\beta b_{k+1,1} - (1 - \beta + k\alpha)b_{k,1} + (k-1)\alpha b_{k-1,1} = 0. \tag{3.11}$$

In the following we will solve (3.11) by using the Laplace Method. This method was first used in the study of web graph models by [9]. Replacing k by $k+1$ in (3.11), we get

$$\beta b_{k+2,1} + [-\alpha(k+1) + \beta - 1]b_{k+1,1} + k\alpha b_{k,1} = 0,$$

which is of the form

$$(A_2(k+2) + B_2)b_{k+2,1} + (A_1(k+1) + B_1)b_{k+1,1} + (A_0 k + B_0)b_{k,1} = 0. \tag{3.12}$$

where $A_2 = 0, B_2 = \beta, A_1 = -\alpha, B_1 = \beta - 1, A_0 = \alpha, B_0 = 0$. We make the substitution

$$b_{k,1} = \int_a^b t^{k-1} v(t)dt, \tag{3.13}$$

where a, b are constants, and $v(t)$ is a function of t to be determined. Integrating by parts, we obtain that

$$kb_{k,1} = [t^k v(t)]_a^b - \int_a^b t^k v'(t)dt. \tag{3.14}$$

Let $\phi_1(t) = A_2t^2 + A_1t + A_0$ and $\phi_0(t) = B_2t^2 + B_1t + B_0$. Substituting (3.13) and (3.14) into (3.12), we can get that

$$[t^k\phi_1(t)v(t)]_a^b - \int_a^b t^k\phi_1(t)v'(t)dt + \int_a^b t^{k-1}\phi_0(t)v(t)dt = 0. \tag{3.15}$$

If we ensure that

$$\frac{v'(t)}{v(t)} = \frac{\phi_0(t)}{t\phi_1(t)}, \tag{3.16}$$

and

$$[t^k v(t)\phi_1(t)]_a^b = 0, \tag{3.17}$$

then (3.12) will be satisfied. Now (3.17) can be satisfied by choosing $a = 0$ and b equal to a root of $v(t)\phi_1(t) = 0$. Moreover, since $A_2 = 0, B_2 = \beta, A_1 = -\alpha, B_1 = \beta - 1, A_0 = \alpha, B_0 = 0$, we can obtain that

$$\phi_1(t) = A_2t^2 + A_1t + A_0 = (t-1)(-\alpha)$$

and

$$\phi_0(t) = B_2t^2 + B_1t + B_0 = \beta t^2 + (\beta - 1)t.$$

Thus, we have the following differential equation.

$$\frac{v'(t)}{v(t)} = \frac{\phi_0(t)}{t\phi_1(t)} = \frac{\beta t + \beta - 1}{\alpha(1-t)}. \tag{3.18}$$

Integrating (3.18), we obtain that

$$v(t) = Ce^{-\frac{\beta}{\alpha}t}(1-t)^{\gamma}.$$

where $\gamma = \frac{1-2\beta}{\alpha}$ and C is a constant. For convenience, we choose $C = 1$. With this choice of $v(t)$, we can choose $b = 1$ and (3.17) is satisfied. So, we have $a = 0, b = 1$ and $v(t) = e^{-\frac{\beta}{\alpha}t}(1-t)^{\gamma}$.

Now we go back to (3.13) and determine b_k as follows.

$$\begin{aligned}
b_{k,1} &= \int_0^1 t^{k-1}v(t)dt \\
&= \int_0^1 t^{k-1}e^{-\frac{\beta}{\alpha}t}(1-t)^{\gamma}dt \\
&= \int_0^1 t^{k-1}(1-t)^{\gamma}\sum_{j=0}^{\infty}\frac{(-\frac{\beta}{\alpha}t)^j}{j!}dt \\
&= \sum_{j=0}^{\infty}\frac{(-\frac{\beta}{\alpha})^j}{j!}\int_0^1 t^{k+j-1}(1-t)^{\gamma}dt \\
&= \sum_{j=0}^{\infty}\frac{(-\frac{\beta}{\alpha})^j}{j!}\frac{\Gamma(k+j)\Gamma(\gamma+1)}{\Gamma(k+j+\gamma+1)} \\
&= \sum_{j=0}^{\infty}\frac{(-\frac{\beta}{\alpha})^j\Gamma(\gamma+1)}{j!}\frac{\Gamma(k+j)}{\Gamma(k+j+\gamma+1)}.
\end{aligned}$$

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. We write $a_n \approx b_n$ if

$$\lim_{n\to\infty}\frac{a_n}{b_n} = 1.$$

Assuming that k is large, then we can use Stirling's formula for $\Gamma(k+j)$ and $\Gamma(k+j+\gamma+1)$ as follows.

$$\Gamma(k+j) \approx \sqrt{2\pi}(k+j-1)^{k+j-\frac{1}{2}}e^{-(k+j-1)}$$

and

$$\Gamma(k+j+\gamma+1) \approx \sqrt{2\pi}(k+j+\gamma)^{k+j+\gamma+\frac{1}{2}}e^{-(k+j+\gamma)}$$

Hence, we obtain that

$$\begin{aligned}
b_{k,1} &= \sum_{j=0}^{\infty}\frac{(-\frac{\beta}{\alpha})^j\Gamma(\gamma+1)}{j!}\frac{\Gamma(k+j)}{\Gamma(k+j+\gamma+1)} \\
&= (1+O(k^{-1}))\sum_{j=0}^{\infty}\frac{e^{1+\gamma}(-\frac{\beta}{\alpha})^j\Gamma(\gamma+1)}{j!}(k+\gamma+j)^{-\gamma-1} \\
&= (1+O(k^{-1}))C_1(\alpha,\beta)k^{-\gamma-1},
\end{aligned}$$

where $C_1(\alpha,\beta)$ is a constant.

The sequence of random variables $\{d_{k,t}^{in}\}$ satisfies the 1-Lipschitz condition. By Lemma 2, for every $\epsilon > 0$, we obtain that

$$Pr(|d_{k,t}^{in} - b_{k,1}t| > 2\epsilon\sqrt{t}) < 2e^{-\epsilon^2/2}, \tag{3.19}$$

By Lemma 4, for every $\epsilon > 0$, we have that

$$Pr(|n_t - n_0 - (1-\alpha-\beta)t| > \epsilon\sqrt{t}) < 2e^{-\frac{\epsilon^2}{2}}.$$

With t large enough so that $n_0 \leq \epsilon\sqrt{t}$, thus we obtain that

$$Pr(|n_t - (1-\alpha-\beta)t| > 2\epsilon\sqrt{t}) < 2e^{-\frac{\epsilon^2}{2}}. \tag{3.20}$$

By (3.19) and (3.20), we obtain that

$$Pr\left(\left|d_{k,t}^{in} - \frac{b_{k,1}}{1-\alpha-\beta}n_t\right| > 2\epsilon\sqrt{t}\left(1 + \frac{b_{k,1}}{1-\alpha-\beta}\right)\right) < 4e^{-\epsilon^2/2},$$

The proof of (1) follows. ▮

4 Conclusion and discussion

In this paper, we use techniques in random graph theory to analyze power law graphs, and we solve recurrences by the Laplace method. Our models generate graphs with power law exponent in the interval $(1, \infty)$. This is a larger interval than the interval $[2, \infty)$ found for the models in [7, 9]. This is significant since certain massive networks, such as the network of protein-protein interaction networks in a living cell, have power law exponents in the interval $(1, 2)$; see [8]. Hence, our models can be used not only as models of the web graph, but for many other massive networks.

A problem that we cannot presently solve is how to rigorously analyze models that incorporate the deletion of nodes over time. For instance, one might want to make the deletion of high degree nodes less likely than low degree nodes. We expect to pursue this problem in future work.

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