Chapter 8

Tests of Hypotheses Based on a Single Sample

8.1 Hypotheses and Test Procedures

Hypotheses

The *null hypothesis*, denoted H_0, is the claim that is initially assumed to be true. The *alternative hypothesis*, denoted by H_a, is the assertion that is contrary to H_0. Possible conclusions from hypothesis-testing analysis are *reject* H_0 or *fail to reject* H_0.

A Test of Hypotheses

A *test of hypotheses* is a method for using sample data to decide whether the null hypothesis should be rejected.

Test Procedure

A test procedure is specified by

1. A *test statistic*, a function of the sample data on which the decision is to be based.
2. A *rejection region*, the set of all test statistic values for which H_0 will be rejected (null hypothesis rejected iff the test statistic value falls in this region.)

Errors in Hypothesis Testing

A *type I error* consists of rejecting the null hypothesis H_0 when it was true.

A *type II error* involves not rejecting H_0 when H_0 is false.

Rejection Region: α and β

Suppose an experiment and a sample size are fixed, and a test statistic is chosen. The decreasing the size of the rejection region to obtain a smaller value of α results in a larger value of β for any particular parameter value consistent with H_a.

Significance Level

Specify the largest value of α that can be tolerated and find a rejection region having that value of α. This makes β as small as possible subject to the bound on α. The resulting value of α is referred to as the *significance level*.

Level α Test

A test corresponding to the significance level is called a *level α test*. A test with significance level α is one for which the type I error probability is controlled at the specified level.

8.2 Tests About a Population Mean

Case I: A Normal Population With Known σ

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic value: $z = \dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$

Case I: A Normal Population With Known σ

Alternative Hypothesis	Rejection Region for Level α Test
$H_a: \mu > \mu_0$	$z \geq z_\alpha$
$H_a: \mu < \mu_0$	$z \leq -z_\alpha$
$H_a: \mu \neq \mu_0$	$z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$

Recommended Steps in Hypothesis-Testing Analysis

1. Identify the parameter of interest and describe it in the context of the problem situation.
2. Determine the null value and state the null hypothesis.
3. State the alternative hypothesis.

Hypothesis-Testing Analysis

4. Give the formula for the computed value of the test statistic.

5.State the rejection region for the selected significance level

6. Compute any necessary sample quantities, substitute into the formula for the test statistic value, and compute that value.

Hypothesis-Testing Analysis

7. Decide whether H_0 should be rejected and state this conclusion in the problem context.

The formulation of hypotheses (steps 2 and 3) should be done before examining the data.

Type II Probability $\beta(\mu')$ for a Level α Test

Alt. Hypothesis	Type II Probability $\beta(\mu')$
$H_a: \mu > \mu_0$	$\Phi\left(z_\alpha + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
$H_a: \mu < \mu_0$	$1 - \Phi\left(-z_\alpha + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
$H_a: \mu \neq \mu_0$	$\Phi\left(z_{\alpha/2} + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\alpha/2} + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

Sample Size

The sample size n for which a level α test also has $\beta(\mu') = \beta$ at the alternative value μ' is

$$n = \begin{cases} \left[\dfrac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{one-tailed test} \\ \left[\dfrac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{two-tailed test} \end{cases}$$

Case II: Large-Sample Tests

When the sample size is large, the z tests for case I are modified to yield valid test procedures without requiring either a normal population distribution or a known σ.

Large Sample Tests ($n > 40$)

For large n, s is close to σ.

Test Statistic: $Z = \dfrac{\overline{X} - \mu_0}{S / \sqrt{n}}$

The use of rejection regions for case I results in a test procedure for which the significance level is approximately α.

Case III: A Normal Population Distribution

If $X_1, \ldots, X_n$ is a random sample from a normal distribution, the standardized variable

$$T = \frac{\overline{X} - \mu}{S / \sqrt{n}}$$

has a t distribution with $n - 1$ degrees of freedom.

The One-Sample t Test

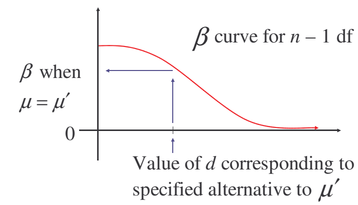
Null hypothesis: $H_0 : \mu = \mu_0$

Test statistic value: $t = \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}}$

The One-Sample t Test

Alternative Hypothesis	Rejection Region for Level α Test
$H_a : \mu > \mu_0$	$t \ge t_{\alpha, n-1}$
$H_a : \mu < \mu_0$	$t \le -t_{\alpha, n-1}$
$H_a : \mu \ne \mu_0$	$t \ge t_{\alpha/2, n-1}$ or $t \le -t_{\alpha/2, n-1}$

A Typical β Curve for the t Test



8.3

Tests Concerning a Population Proportion

A Population Proportion

Let p denote the proportion of individuals or objects in a population who possess a specified property.

Large-Sample Tests

Large-sample tests concerning p are a special case of the more general large-sample procedures for a parameter θ.

Large-Samples Concerning p

Null hypothesis: $H_0 : p = p_0$

Test statistic value:

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

Large-Samples Concerning p

Alternative Hypothesis	Rejection Region
$H_a : p > p_0$	$z \ge z_\alpha$
$H_a : p < p_0$	$z \le -z_\alpha$
$H_a : p \ne p_0$	$z \ge z_{\alpha/2}$ or $z \le -z_{\alpha/2}$

Valid provided $np_0 \ge 10$ and $n(1 - p_0) \ge 10$.

General Expressions for $\beta(p')$

Alt. Hypothesis	$\beta(p')$
$H_a : p > p_0$	$\Phi\left(\dfrac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$
$H_a : p < p_0$	$1 - \Phi\left(\dfrac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$

General Expressions for $\beta(p')$

Alt. Hypothesis	$\beta(p')$
$H_a : p \ne p_0$	$\Phi\left(\dfrac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right) - \Phi\left(\dfrac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$

Sample Size

The sample size n for which a level α test also has $\beta(p') = p$

$$n = \begin{cases} \left[\dfrac{z_\alpha\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 & \text{one-tailed test} \\ \left[\dfrac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 & \text{two-tailed test} \end{cases}$$

Small-Sample Tests

Test procedures when the sample size n is small are based directly on the binomial distribution rather than the normal approximation.

$$P(\text{type I}) = 1 - B(c - 1; n, p_0)$$

$$B(p') = B(c - 1; n, p')$$

8.4

P - Values

P - Value

The *P-value* is the smallest level of significance at which H_0 would be rejected when a specified test procedure is used on a given data set.

1. P-value $\le \alpha$

   $\Rightarrow$ reject H_0 at a level of α
2. P-value $> \alpha$

   $\Rightarrow$ do not reject H_0 at a level of α

P - Value

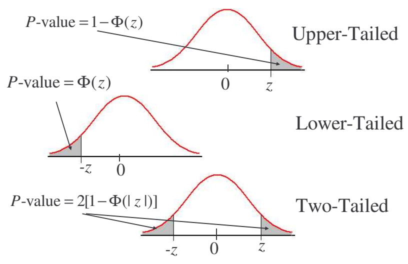
The *P-value* is the probability, calculated assuming H_0 is true, of obtaining a test statistic value at least as contradictory to H_0 as the value that actually resulted. The smaller the P-value, the more contradictory is the data to H_0.

P-Values for a z Test

P-value:

$$P = \begin{cases} 1-\Phi(z) & \text{upper-tailed test} \\ \Phi(z) & \text{lower-tailed test} \\ 2\left[1-\Phi(|z|)\right] & \text{two-tailed test} \end{cases}$$

P-Value (area)



P–Values for t Tests

The P-value for a t test will be a t curve area. The number of df for the one-sample t test is $n - 1$.

8.5

Some Comments on Selecting a Test Procedure

Constructing a Test Procedure

1. Specify a test statistic.
2. Decide on the general form of the rejection region.
3. Select the specific numerical critical value or values that will separate the rejection region from the acceptance region.

Issues to be Considered

1. What are the practical implications and consequences of choosing a particular level of significance once the other aspects of a test procedure have been determined?
2. Does there exist a general principle that can be used to obtain best or good test procedures?

Issues to be Considered

1. When there exist two or more tests that are appropriate in a given situation, how can the tests be compared to decide which should be used?
2. If a test is derived under specific assumptions about the distribution of the population being sampled, how well will the test procedure work when the assumptions are violated?

Statistical Versus Practical Significance

Be careful in interpreting evidence when the sample size is large, since any small departure from H_0 will almost surely be detected by a test (*statistical significance*), yet such a departure may have little *practical significance*.

The Likelihood Ratio Principle

1. Find the largest value of the likelihood for any θ in Ω_0.
2. Find the largest value of the likelihood for any θ in Ω_a.
3. Form the ratio

$$\lambda(x_1,\ldots,x_n) = \frac{\text{maximum likelihood for } \theta \text{ in } \Omega_0}{\text{maximum likelihood for } \theta \text{ in } \Omega_a}$$

Reject H_0 when this ratio is small.