Math 2790: Assignment 2

Due on Thursday, October 18^{th}

(Formative Evaluation due on Tuesday, October 16^{th})

SECTION A: you may answer *either* the first five questions *or* the sixth question. Each of the first five questions will be marked out of 5. The sixth question will be marked out of 25.

1. Determine the last two digits of 7^{2003}.

2. Pick any five elements from the set $\{1, 2, 3, \ldots, 2000, 2001\}$. Prove that regardless of which five elements we pick, we can always select three of those (five) integers such that their sum is divisible by 3. (For example, if we pick the five integers $14, 57, 135, 892, 1732$, then the sum of 14, 57, and 892 is divisible by 3).

3. Determine all positive integers n for which the number $3^n + 2 \cdot 7^n$ is a perfect square.

4. There are x books in the Old Testament, where x is a two-digit integer. If you multiply the digits of x you get the integer y, which is the number of books in the New Testament. Adding x and y, we get 66, the number of books in the Bible. Determine x and y.

5. (a) Find all pairs of positive integers (x, y) such that $2^x + 1 = y^2$.

   (b) Find all pairs of positive integers (x, y) such that $2^x - 1 = y^2$.

6. For each pair of positive integers a and b with $a < b$, define $f(a, b)$ to be the sum of the integers from a to b inclusive. For example, $f(2, 7) = 2 + 3 + 4 + 5 + 6 + 7 = 27$. Define an *O'Shino pair* to be an ordered pair of integers (a, b), with $a < b$ such that $f(a, b)$ equals the digits of a followed by the digits of b. For example, $(2, 7)$ is an O'Shino pair, since $f(2, 7) = 27$. Another O'Shino pair is $(18, 63)$ because $f(18, 63) = 18 + 19 + 20 + \ldots + 62 + 63 = 1863$.

   (a) Find all O'Shino pairs (a, b) where b has exactly one digit.

   (b) Show that $(13, 53)$, $(133, 533)$, $(1333, 5333)$, $(13333, 53333), \ldots$ are all O'Shino pairs.

   (c) Investigate this idea further and see how far you can take it!

   (Note: to get the full twenty-five marks, you do NOT have to completely solve the problem, i.e., characterize all O'Shino pairs. I am more interested in your ideas and progress than anything else.)

SECTION B: Answer all three of these questions. Each of these questions will be marked out of 5.

7. A number is called *multiplicatively perfect* if it is equal to the *product* of its proper divisors. For example, 10 is a multiplicatively perfect number because $10 = 1 \times 2 \times 5$. Determine all multiplicatively perfect numbers and prove that there can be no others.

8. Karin, Chris, and Sable participated in the "Mathletic Games", in which there were M different events. In each event p points were awarded for first place, q points were awarded for second place, and r points were awarded for third place, where $p > q > r > 0$, and p, q, r are integers. Karin finished with 22 points, and Chris and Sable both finished with 9 points. Sable won the shot put. What is the value of M, and who finished second in the high jump?

9. If integers (a, b, c), with $\gcd(a, b, c) = 1$ satisfy the equation $a^2 + b^2 = c^2$, we say that (a, b, c) is a *primitive Pythagorean triple*. Prove that every primitive Pythagorean triple must be of the form $(p^2 - q^2, 2pq, p^2 + q^2)$, where p and q are positive integers.

   (For example, $(5, 12, 13)$ *is a primitive Pythagorean triple and for this triple we have* $p = 3$ *and* $q = 2$*.)*