Solutions To Assignment 2

1. Since $7^4 \equiv 1 \pmod{100}$, we have $7^{2000} = (7^4)^{500} \equiv 1^{500} \equiv 1 \pmod{100}$.

   So we have $7^{2003} = 7^3 \cdot 7^{2000} \equiv 343 \cdot 1 \equiv 43 \pmod{100}$.

   Therefore, the last two digits of 7^{2003} are 43.

2. We pick five elements arbitrarily from the set $\{1, 2, 3, \ldots, 2000, 2001\}$. Call this set S. I f three of these five elements from S, say a, b, and c, are congruent modulo 3, then their sum will be divisible by 3. This follows since $a \equiv b \equiv c \pmod 3$ implies that $a + b + c \equiv c + c + c = 3c \equiv 0 \pmod 3$.

   So suppose that no three of these five elements are congruent modulo 3, as otherwise we are done. Assume that there is no element in S that is congruent to 0 modulo 3. Then all five elements are congruent to 1 or 2 modulo 3. We have five elements and two possible congruence classes, and so by the Pigeonhole Principle, three of these elements must be congruent modulo 3, which is a contradiction. Thus, there must be at least one element in S (call it p) that is congruent to 0 modulo 3. By the same reasoning, there must be at least one element in S (call it q) congruent to 1 modulo 3, and at least one element in S (call it r) congruent to 2 modulo 3. Then $p + q + r \equiv 0 + 1 + 2 \equiv 0 \pmod 3$.

   Therefore, we have shown that if we have three elements in S that are congruent modulo 3 then the sum of these three elements is divisible by 3. Otherwise, there must exist three elements in S congruent to 0, 1, and 2 modulo 3, and the sum of these three elements will be divisible by 3.

3. *Solution 1:* Let $P_n = 3^n + 2 \times 7^n$, for each non-negative integer n. Let's list the first few values for P_i, and see if we notice any patterns. $P_1 = 17$, $P_2 = 107$, $P_3 = 713$, $P_4 = 4883$, $P_5 = 33857$, $P_6 = 236027$, $P_7 = 1649273$. It appears that every term ends in either a 3 or a 7. This motivates us to analyze the problem in mod 10, because we have a strong suspicion that P_n must be congruent to 3 or 7 (mod 10), for every non-negative integer n. So let us try to prove this.

   3^n is congruent to 3, 9, 7, or 1 (mod 10), depending on the value of n. If we make a list of powers of 3 (beginning with 3^0), and look at each of their remainders upon division by ten (i.e., looking at the last digit), we will notice a cycle: $1, 3, 9, 7, 1, 3, 9, 7, 1, \ldots$. The pattern repeats since $3^4 \equiv 1 \pmod{10}$, and so $3^{k+4} \equiv 3^k \pmod{10}$ for each integer $k \geq 0$. Thus, there are four cases to consider:

   If $n \equiv 0 \pmod 4$, then $3^n \equiv 1 \pmod{10}$.

   If $n \equiv 1 \pmod 4$, then $3^n \equiv 3 \pmod{10}$.

   If $n \equiv 2 \pmod 4$, then $3^n \equiv 9 \pmod{10}$.

   If $n \equiv 3 \pmod 4$, then $3^n \equiv 7 \pmod{10}$.

   We do the same thing with 2×7^n. We notice that:

   If $n \equiv 0 \pmod 4$, then $2 \times 7^n \equiv 2 \pmod{10}$.

   If $n \equiv 1 \pmod 4$, then $2 \times 7^n \equiv 4 \pmod{10}$.

If $n \equiv 2 \pmod 4$, then $2 \times 7^n \equiv 8 \pmod{10}$.

If $n \equiv 3 \pmod 4$, then $2 \times 7^n \equiv 6 \pmod{10}$.

Thus, combining these two results, we find that:

If $n \equiv 0 \pmod 4$, then $3^n + 2 \times 7^n \equiv 1 + 2 \equiv 3 \pmod{10}$.

If $n \equiv 1 \pmod 4$, then $3^n + 2 \times 7^n \equiv 3 + 4 \equiv 7 \pmod{10}$.

If $n \equiv 2 \pmod 4$, then $3^n + 2 \times 7^n \equiv 9 + 8 \equiv 7 \pmod{10}$.

If $n \equiv 3 \pmod 4$, then $3^n + 2 \times 7^n \equiv 7 + 6 \equiv 3 \pmod{10}$.

Hence, we have proven that $3^n + 2 \times 7^n$ must be congruent to 3 or 7 (mod 10), for each non-negative integer n. However, the quadratic residues modulo 10 are 0, 1, 4, 5, 6, and 9, and so if a number is congruent to 3 or 7 (mod 10), then the number cannot be a perfect square. Hence, we conclude that there is no non-negative integer n for which $3^n + 2 \times 7^n$ is a perfect square.

Solution 2: (Ian Brown) This is brilliant. Let's analyze modulo 3. Then $P_n = 3^n + 2 \times 7^n \equiv 0^n + 2 \times 1^n = 2$. In other words, every P_n must be congruent to 2 modulo 3. However, the set of quadratic residues modulo 3 is $\{0, 1\}$. Thus, no P_n can be a perfect square, and so we're done.

4. Since x is a two-digit number, let $x = 10a + b$, where $1 \leq a \leq 9$ and $0 \leq b \leq 9$. Then $y = ab$, since it is the product of the digits of x. We are given that $x + y = 10a + b + ab = 66$.

   Solution 1: Adding 10 to both sides and factoring, we have:

$$\begin{aligned} 10a + b + ab &= 66 \\ 10a + b + ab + 10 &= 76 \\ a(b + 10) + (b + 10) &= 76 \\ (a + 1)(b + 10) &= 76 \end{aligned}$$

   Since b is a digit, $b + 10$ must be between 10 and 19. The only divisor of 76 in this range is 19, and so $b + 10$ must be 19. Thus, $a + 1$ must be 4. Solving, we get $a = 3$ and $b = 9$, and so $x = 39$. Hence, $y = 27$.

   Solution 2: Solving for b in terms of a, we have:

$$\begin{aligned} 10a + b + ab &= 66 \\ b(a + 1) + 10a &= 66 \\ b(a + 1) &= 66 - 10a = 76 - (10a + 10) \\ b &= \frac{76}{a+1} - \frac{10a + 10}{a + 1} \\ b &= \frac{76}{a+1} - 10 \end{aligned}$$

   Since a and b are both integers, $\frac{76}{a+1}$ must be an integer. Thus, $a + 1$ must divide $76 = 2^2 \times 19$. Since a is a digit, a must be between 1 and 10. So $a + 1$ is either 2 or

2^2. If $a+1=2$, then $a=1$ and $b=\frac{76}{2}-10=28$, which is not a digit. So the only possibility is to have $a+1=4$, so $a=3$ and $b=\frac{76}{4}-10=9$. Thus, $x=39$ and $y=27$.

5. a) $2^x+1=y^2$ can be rewritten as $2^x=y^2-1=(y+1)(y-1)$. So we have two terms, $y+1$ and $y-1$, which multiply to give a power of 2. That means that both $y+1$ and $y-1$ must itself be powers of 2, for if either term contained a prime divisor other than 2, then $(y+1)(y-1)$ could not possibly be a power of 2. So we are looking for two powers of 2 that differ by 2 (since $(y+1)-(y-1)=2$), and clearly the only possibility is to have $y+1=2^2=4$ and $y-1=2^1=2$. Thus, y must be 3, and $2^x=8$, so $x=3$. Therefore, the only solution is $(x,y)=(3,3)$.

   b) Let us analyze modulo 4. If $x\geq 2$, then $2^x=2^2\cdot 2^{x-2}$ is a multiple of 4, and so $2^x\equiv 0 \pmod 4$. Thus, if $x\geq 2$, we must have $y^2=2^x-1\equiv 0-1\equiv 3 \pmod 4$. But 3 is not a quadratic residue modulo 4, and so there is no integer y for which $y^2\equiv 3 \pmod 4$. So if $x\geq 2$, the equation has no integer solutions. So we only need to try $x=1$. In this case, we easily see that the solution is $(x,y)=(1,1)$, since y must be a positive integer. Thus, this is the only solution.

6. This is the O'Shino pairs question. Check out my article, which can be found from the webpage.

7. If n is prime, then its only proper divisor is 1, and so n cannot be multiplicatively perfect. So n must have at least one proper divisor p, with $p\neq 1$. Well, if p is a divisor of n, then $n=p\times\frac{n}{p}$, where $\frac{n}{p}$ is an integer. So $\frac{n}{p}$ must be a divisor of n as well. Note that if $n=p^2$, $\frac{n}{p}=p$, so we must consider this case separately. In this special case, if p is prime, then the only divisors of $n=p^2$ are 1 and p, and so n is not multiplicatively perfect, and if p is not a prime, then $n=p^2$ must have more than three proper divisors and we will show later on that this condition guarantees that n cannot be multiplicatively perfect. So suppose that $n\neq p^2$.

   Note that the product of p and $\frac{n}{p}$ is n, and so if n has any other proper divisors other than 1, p, and $\frac{n}{p}$, then its product will exceed n and not be perfect. Thus, in order for n to be perfect it must have exactly 3 proper divisors, namely 1, p, and $\frac{n}{p}$. If n has more than 3 proper divisors, then the product of the proper divisors will exceed n.

   To find which numbers have 3 proper divisors, i.e., 4 divisors, let $n=p_1^{a_1}p_2^{a_2}p_3^{a_3}\cdots p_k^{a_k}$. Then the number of divisors of n is $(a_1+1)(a_2+1)(a_3+1)\cdots(a_k+1)$. Assume without loss of generality that $a_1\geq a_2\geq a_3\geq\ldots\geq a_k$. So $(a_1+1)(a_2+1)(a_3+1)\cdots(a_k+1)=4$ gives us only two cases:

   i) $a_1+1=4$, and $a_i+1=1$ for every other i.

   ii) $a_1+1=2$, $a_2+1=2$, and $a_i+1=1$ for every other i.

   In the former case, we have $a_1=3$, so n must be of the form p^3, where p is prime. In the latter case, we have $a_1=a_2=1$, and so n must be of the form $p^1q^1=pq$, where p and q are distinct primes.

   Thus, there are only two types of multiplicatively perfect numbers: the numbers that are the cube of a prime (e.g. $2^3=8$), and the numbers that are the product of two distinct primes (e.g. $5 \cdot 7 = 35$).

8. There were M events and in each event, a total of $p+q+r$ points were awarded. Hence, there were $M(p+q+r)$ total points awarded. Since the total number of points was $22+9+9=40$, it follows that $M(p+q+r)=40$.

   Since $p > q > r > 0$, we have $p+q+r \geq 3+2+1 = 6$, and so there are only five cases to consider: $M=1, p+q+r=40$, $M=2, p+q+r=20$, $M=4, p+q+r=10$, and $M=5, p+q+r=8$.

   We can immediately reject the first case. If $M=2$, then Karin got at most $p+q$ points, because she came second in one of the two events. But $p+q < p+q+r = 20 < 22$, so she could not have possibly gotten 22 points. So we can reject this case too.

   If $M=4$, then $p+q+r=10$. The possible values for (p,q,r) are $(7,2,1)$, $(6,3,1)$, $(5,4,1)$, and $(5,3,2)$. For $(p,q,r)=(7,2,1)$, the worst Sable could have done was to come last in the rest of her events, so she must have at least $p+3r=10$ points. However, she got 9 points. For the other three cases for $M=4$, the best Karin could have done was $3p+q$, since she did not win the shot put. But for each of $(6,3,1)$, $(5,4,1)$, and $(5,3,2)$, we have $3p+q<22$. Thus, we must reject these cases as well. So $M \neq 4$.

   That leaves us with $M=5$ as the only possibility. Since $p+q+r=8$, we have either $(p,q,r)=(4,3,1)$ or $(p,q,r)=(5,2,1)$. In the first case, the best Karin could have done was finish with $4p+q=19$ points, so she could not have possibly gotten 22 points. Hence, we must have $p=5, q=2, r=1$. The best Karin could have done was score $4p+q=22$ points, which is exactly what she got. So Karin must have won every event other than the shot put. The worst Sable could have done was score $p+4r=9$ points, which is exactly what she got. So Sable must have came third in every event other than the shot put. Hence, it follows that Chris must have come second in every event other than the shot put (indeed, if he finished last in the shot put and second in everything else, he got $4q+r=9$ points).

   Thus, we conclude that $M=5$, and Chris finished second in the high jump.

9. Suppose that a and b are both even. Then $c^2=a^2+b^2$ is also even, so c is even. This contradicts the fact that $\gcd(a,b,c)=1$.

   Suppose that a and b are both odd. Then $c^2=a^2+b^2 \equiv 1+1 \equiv 2 \pmod 4$. But the quadratic residues modulo 4 are 0 and 1, and so c^2 cannot possibly be congruent to 2 modulo 4. We have a contradiction.

   Hence a and b must have opposite parity. Without loss of generality, assume b is even. Let $b=2k$, for some positive integer k. Then $c^2-a^2=b^2=4k^2$. Now, a is odd, and so $c^2=a^2+b^2$ is odd as well. Thus, c is odd. Since a and c are both odd, it follows that $c+a$ and $c-a$ are both even. So $c+a=2m$ and $c-a=2n$ for some positive integers m and n. (Note: $c>a$). Thus, $4mn=(c+a)(c-a)=c^2-a^2=4k^2$, so $mn=k^2$. We will show that m and n must both be perfect squares.

   Suppose that $\gcd(m,n) \neq 1$. Then m and n are both multiples of d, where d is some prime number. If we solve for c and a in the equations above, we get $c=m+n$ and

$a = m - n$. Hence, c and a are both multiples of d since m and n both are. Finally, $b^2 = mn$, so b^2 must be a multiple of d^2, since m and n are both multiples of d. That implies that b is a multiple of d. So a, b, and c are all multiples of d, and this contradicts the fact that $\gcd(a, b, c) = 1$.

Thus, we must have $\gcd(m, n) = 1$. Let $k = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_j^{a_j}$. Then $mn = k^2 = p_1^{2a_1} p_2^{2a_2} p_3^{2a_3} \cdots p_j^{2a_j}$. Suppose m is not a perfect square. Then if we look at the prime factorization of m, at least one of the exponents will *not* be an even number. (e.g. $2^2 \cdot 3^6 \cdot 5^3$ is not a perfect square). We will find a contradction.

Say that p_1^{2j-1} is the highest power of p_1 that divides m, where j is some positive integer less than a_1. In other words, p_1^{2j} does *not* divide m. Since $p_1^{2a_1}$ divides mn, it follows that $p_1^{2a_1-(2j-1)} = p_1^{2a_1-2j+1}$ must divide n. But $2a_1 - 2j + 1$ is odd, and so it is at least 1. That means, p_1 divides m, and p_1 divides n, and that contradicts the fact that $\gcd(m, n) = 1$. Hence, if we look at the prime factorization of m, every prime number will be raised to an even exponent. Thus, m must be a perfect square. By symmetry, n must be a perfect square as well.

Hence, $m = p^2$ and $n = q^2$ for some positive integers p and q. Since $k^2 = mn$, we have $k = \sqrt{mn} = \sqrt{p^2q^2} = pq$, and so $b = 2k = 2pq$. Since $a = m - n$ and $c = m + n$, we have $a = p^2 - q^2$, and $c = p^2 + q^2$. Thus, we have proven that every primitive Pythagorean triple (a, b, c) must be of the form $(p^2 - q^2, 2pq, p^2 + q^2)$, where p and q are positive integers.